NUMERICAL CALCULATIONS OF THE MEW EQUATION FROM A NEW PERSPECTIVE

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Abstract: Numerical computations for natural systems and acquiring travelling wave solutions of nonlinear wave equations in relation to sciences such as optics, fluid mechanics, solid state physics, plasma physics, kinetics, and geology have become very important in the field of mathematical modeling recently. For this, many methods have been suggested. The strategy applied for this article is to obtain more perfect numerical solutions of Modified Equal Width equation (MEW), which is one of the equations used to model the nonlinear phenomena mentioned. For this purpose, the Lie-Trotter splitting technique is applied to the MEW equation. Firstly, the problem is split into two sub-problems, one linear and the other nonlinear, containing derivative with respect to time. Secondly, each subproblem is reduced to the algebraic equation system by using collocation finite element method (FEM) based on the quintic B-spline approximate functions for spatial discretization and the convenient classical finite difference approaches for temporal discretization. Then, the obtained systems are solved with the Lie Trotter splitting algorithm. Explanatory test problems are considered, showing that the newly proposed algorithm has superior accuracy than previous methods, and the numerical results produced by the proposed algorithm are shown in tables and graphs. In addition, the stability analysis of the new approach is examined. Therefore, it is appropriate to state that this new technique can be easily applied to partial differential equations used in other disciplines in terms of the results obtained and the cost of Matlab calculation software.

1 Introduction

The MEW equation with physical boundary conditions $U \to 0$ when $x \to \pm\infty$ associated with the EW equation introduced by Morrison *et al.* [2], used for modeling nonlinear disperse wave events, is given in the following form

$$U_t + 3U^2 U_x - \mu U_{xxt} = 0 \tag{1}$$

where μ is a non-negative constant, $U(x,t)$ is the wave amplitude and x and t indicate spatial and temporal variables subindices. Here, it is very important to state that solitary waves are known as traveling waves in the literature retained their shapes and speeds because of the sensitive balance between nonlinearity and dispersive effects, whereas a soliton is a very private type of solitary waves, retaining its shape and speed even after colliding with another wave. When examining the literature, one can see that many authors have obtained both analytical and numerical solutions for the modified equal width equation until present time. Some of them can be listed as: Wang *et al.* [6] used the method of dynamical system for the exact travelling wave solutions of the MEW equation. Wazwaz [3] studied on a sine–cosine ansatz and the tanh method. Jin [4] proposed an analytical approach based on the homotopy perturbation method for equation. Hamdi *et al.* [1] derived exact solitary wave solutions for the generalized EW and EW-Burgers equation. Lu [5] presented variational iteration method for analytical solution of the equation. Taghizadeh [7] applied the modified simple equation method. Taha and Noorani [8] developed the G'/G-expansion method. Rui *et al.* [9] used integral bifurcation method. Cheng and Liew [10] actualized an improved element-free Galerkin (IEFG) method for numerical solution of equation. Both Esen [11] and Karakoç and Geyikli [12] implemented Lumped Galerkin method combined with quadratic and cubic B-splines

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respectively. Gardner and Gardner [13] solved with Galerkin's method to the EW equation. Esen and Kutluay [14] and Raslan *et al.* [15] utilized finite difference method. Essa [16] worked the multigred method. Geyikli and Karakoç [17] and Karakoç and Geyikli [18] investigated the approximate solutions of the equation by Subdomain finite element method using both quartic and sextic B-splines respectively. While Geyikli and Karakoç [19] solved the MEW equation with the Petrov Galerkin method, Roshan [20] solved the GEW equation with the same method. Evans and Raslan [21] submitted a collocation method with quadratic B-splines for the GEW equation. Dereli[22] researched by utilizing meshless method based on collocation with the well-known radial basis functions. Başhan *et al.* [23] worked the finite difference method combined with differential quadrature method. Karakoç *et al.* [24] used different linearization techniques with the help of cubic B-spline collocation FEM. Çelikkaya [25] solved the equation with Strang splitting scheme using collocation method with the cubic B-spline. Zaki [26] studied the MEW equation by Petrov Galerkin method with quintic B-spline. Saka [27] implemented collocation algorithms with Quintic B-spline. Additionally, as the work of recent years, Başhan *et al.* [28] have submitted a new perspective for equation. Yağmurlu and Karakaş [29] have proposed Trigonometric cubic B-spline method.

In the present article, the Lie-Trotter splitting algorithm is applied by using the collocation finite element method combined with the quintic B-spline to obtain the most perfect results in finding approximate solutions of the MEW equation in line with the works done in recent years with the condition given at initial time

$$U(x,0) = g_0(x), \tag{2}$$

and the conditions given at the boundaries

$$
\begin{aligned}
U(x_L, t) &= U(x_R, t) = 0, \\
U_x(x_L, t) &= U_x(x_R, t) = 0, \\
U_{xx}(x_L, t) &= U_{xx}(x_R, t) = 0 \quad t > 0.
\end{aligned}
\tag{3}
$$

The strategy here is based on the fact that the Lie-Trotter splitting technique along with quintic B-spline base functions is an easier, faster, less costly and successful approach to solving problems such as the MEW equation. The outline of this article can be briefly outlined section by section as follows: First of all, an extensive literature search is made about the studies carried out for the MEW equation until today, and information about its physical properties are given. In section 2, the Lie-Trotter splitting technique is briefly explained. In section 3, the collocation method with the quintic B-spline base functions is introduced and the MEW equation is split into two subequations that each of them includes derivative in the direction of time and are numerically solved with the help of Lie-Trotter splitting algorithm by applying quintic B-spline collocation FEM to both sub-equations. Section 4 explains how to find the initial vector. In section 5, the stability analysis of the numerical scheme obtained by Lie-Trotter splitting technique is examined. In section 6, the approximate results generated by the proposed algorithm and their comparison with tables and graphs are given. At last, section 7 provides comments on the newly generated results and the performance of the numerical algorithm.

2   Time-Splitting Technique

It can sometimes be difficult for a scientist to find an efficient way to do approximate calculations of a complex problem. As a way to achieve this, operator splitting algorithms have emerged, which are based on the concepts of divide and conquer and divided a complex problem into simpler subproblems. There are simple and easy-to-apply methods for each sub-problem that arises. As a result of the application of these methods, approximate schemes are found and these schemes are combined with the operator splitting algorithm, resulting in easy numerical solutions with programs such as Matlab calculation software used in the proposed study. In this study, the Cauchy problem given in the following form will be discussed

$$\frac{dU(t)}{dU} = \hat{A}U(t) + \hat{B}U(t), \ \ U(0) = U_0, \ \ t \in [0, T], \tag{4}$$

in which X is the Banach space and $U_0 \in X$ is the initial function. $\hat{A}, \hat{B}$ can be assumed as bounded linear operators in the space X such that $\hat{A}, \hat{B} : X \to X$. The general forms of first-order diagrams by means of dividing time step Δt with the Lie-Trotter splitting algorithm given in [32] having the schemes $''\hat{A} - \hat{B}''$ and $''\hat{B} - \hat{A}''$ can be presented as follows

$$
\begin{aligned}
\frac{dU^*(t)}{dt} &= \hat{A}U^*(t), \ \ U^*(0) = U_0 \quad over \quad [0, \Delta t], \\
\frac{dU^{**}(t)}{dt} &= \hat{B}U^{**}(t), \ \ U^{**}(0) = U^*(\Delta t) \quad over \quad [0, \Delta t].
\end{aligned}
$$

Here, the final values are acquired by utilizing $U^{**}(\Delta t)$. An error called splitting error occurs when the main problem turns into sub-problems. The local turncation error of this technique is

$$
\begin{aligned}
Te &= \frac{1}{\Delta t}(e^{\Delta t(\hat{A}+\hat{B})} - e^{\Delta t\hat{B}}e^{\Delta t\hat{A}})U(t_n) \\
&= \frac{1}{\Delta t}[\frac{\Delta t^2}{2}(\hat{A}\hat{B} - \hat{B}\hat{A})U(t_n) + O(\Delta t^3)] \\
&= \frac{1}{\Delta t}[\hat{A}, \hat{B})U(t_n) + O(\Delta t^2)]
\end{aligned}
$$

Here $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. Consequently, this indicates to be first-order of present technique [32].

3   Lie-Trotter Splitting Technique with Quintic B-Spline Collocation Method

For the numerical approximation of the MEW equation, the solution region is first confined to the closed interval $[x_L, x_R]$. The interval $[x_L, x_R]$ can be partitioned into finite elements uniformly in terms of knot points x_j such that $x_L = x_0 \leq x_1 \leq ... \leq x_N = x_R, h = x_{j+1} - x_j = \dfrac{x_R - x_L}{N}$ for $j = 0(1)N - 1$. The quintic B-spline base functions $\varphi_j(x)$ at knot points x_j for $j = -2(1)N + 2$ are given as [33]

$$
\varphi_j(x) = \frac{1}{h^5}
\begin{cases}
p_0 = (x - x_{j-3})^5, & x \in [x_{j-3}, x_{j-2}] \\
p_1 = p_0 - 6(x - x_{j-2})^5, & x \in [x_{j-2}, x_{j-1}] \\
p_2 = p_1 - 6(x - x_{j-2})^5 + 15(x - x_{j-1})^5, & x \in [x_{j-1}, x_j] \\
p_3 = p_2 - 6(x - x_{j-2})^5 - 20(x - x_j)^5, & x \in [x_j, x_{j+1}] \\
p_4 = p_3 - 6(x - x_{j-2})^5 + 15(x - x_{j+1})^5, & x \in [x_{j+1}, x_{j+2}] \\
p_5 = p_4 - 6(x - x_{j-2})^5 - 6(x - x_{j+2})^5, & x \in [x_{j+2}, x_{mj+3}] \\
0, & \text{otherwise.}
\end{cases}
\tag{5}
$$

The all of quintic B-spline base functions $\varphi_{-2}(x), \varphi_{-1}(x), \dots, \varphi_{N+2}(x)$ compose a base for the functions on $[x_L, x_R]$. The approximate ones $U_N(x, t)$ corresponding to the exact solutions $U(x, t)$ can be presented in the following form

$$
U_N(x, t) = \sum_{j=-2}^{N+2} \varphi_j(x)\delta_j(t). \tag{6}
$$

Here, the unknown time parameters $\delta_j(t)$ is determined with both boundary and collocation conditions. On a typical element $[x_j, x_{j+1}]$ by the local coordinate transformation described as $h = x - x_j$ for $0 \leq \zeta \leq h$, the quintic B-spline base functions on $[0, h]$ in terms of ζ can be submitted as follows

$$
\begin{aligned}
\varphi_{j-2} &= 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5, \\
\varphi_{j-1} &= 26 - 50\zeta + 20\zeta^2 + 20\zeta^3 - 20\zeta^4 + 5\zeta^5, \\
\varphi_j &= 66 - 60\zeta^2 + 30\zeta^4 - 10\zeta^5, \\
\varphi_{j+1} &= 26 + 50\zeta + 20\zeta^2 - 20\zeta^3 - 20\zeta^4 + 10\zeta^5, \\
\varphi_{j+2} &= 1 + 5\zeta + 10\zeta^2 + 10\zeta^3 + 5\zeta^4 - 5\zeta^5, \\
\varphi_{j+3} &= \zeta^5.
\end{aligned}
\tag{7}
$$

The all of quintic B-spline base functions are zero outside of $\phi_{j-2}, \varphi_{j-1}, \varphi_j, \varphi_{j+1}, \varphi_{j+2}$ and φ_{j+3}. The knot points U_j, U_j', U_j'' are presented in terms of the parameter $\delta_j(t)$ in the following form

$$
\begin{aligned}
U_j &= \delta_{m-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2}, \\
U_j' &= \frac{5}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}), \\
U_j'' &= \frac{20}{h^2}(\delta_{j-2} + 2\delta_{j-1} - 6\delta_j + 2\delta)_{j+1} + \delta_{j+2}),
\end{aligned}
\tag{8}
$$

and the variation of U with the interval $[x_j, x_{j+1}]$

$$U = \sum_{j=-2}^{N+2} \varphi_j \delta_j. \tag{9}$$

Now let the MEW equation be split into to include the derivative with respect to time as follows

$$U_t - U_{xxt} = 0, \tag{10}$$

$$U_t - U_{xxt} + 3U^2 U_x = 0. \tag{11}$$

Now, let's explain the application of the Lie-Trotter division algorithm to the MEW equation: First, equation (10) is solved using initial condition (2). Then this new solution is used as the initial condition of the equation and Equation (11) is solved. From here, the solution of equation (11) is the solution of Equation (1) in the first time step. Thus, approximate solutions at the next time levels are obtained similarly to those in the first time step. Thus, in accordance with the explanation of the Lie–Trotter technique, the solution of the MEW equation with initial and boundary conditions (2) and (3) is acquired by solving two initial boundary value problems (10) and (11) with initial conditions. When the values of nodes U_j, U_j', U_j'' given in (8) are written in (10) and (11) and necessary procedures are applied, the systems of ordinary differential equation presented in the form below are obtained

$$\dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2}) = 0, \tag{12}$$

$$\dot{\delta}_{j-2} + 26\dot{\delta}_{j-1} + 66\dot{\delta}_j + 26\dot{\delta}_{j+1} + \dot{\delta}_{j+2} - \frac{20\mu}{h^2}(\dot{\delta}_{j-2} + 2\dot{\delta}_{j-1} - 6\dot{\delta}_j + 2\dot{\delta}_{j+1} + \dot{\delta}_{j+2})$$
$$+ \frac{5z_j}{h}(-\delta_{j-2} - 10\delta_{j-1} + 10\delta_{j+1} + \delta_{j+2}) = 0, \tag{13}$$

in which the first derivative according to time t is shown with symbol $''.''$ and z_j is gotten as

$$z_j = 3(\delta_{j-2} + 26\delta_{j-1} + 66\delta_j + 26\delta_{j+1} + \delta_{j+2})^2$$

for linearization operation. If $\dfrac{\delta_j^{n+1} + \delta_j^n}{2}$ instead of the parameter δ_j and $\dfrac{\delta_j^{n+1} - \delta_j^n}{\Delta t}$ instead of the parameter $\dot{\delta}_j$ in Eqs.(12) and (13) are written, system of equations given in the following are obtained

$$k_1 \delta_{m-2}^{n+1} + k_2 \delta_{m-1}^{n+1} + k_3 \delta_m^{n+1} + k_4 \delta_{m+1}^{n+1} + k_5 \delta_{m+2}^{n+1} = k_6 \delta_{m-2}^n + k_7 \delta_{m-1}^n + k_8 \delta_m^n + k_9 \delta_{m+1}^n + k_{10} \delta_{m+2}^n \tag{14}$$

$$l_1 \delta_{m-2}^{n+1} + l_2 \delta_{m-1}^{n+1} + l_3 \delta_m^{n+1} + l_4 \delta_{m+1}^{n+1} + l_5 \delta_{m+2}^{n+1} = l_6 \delta_{m-2}^n + l_7 \delta_{m-1}^n + l_8 \delta_m^n + l_9 \delta_{m+1}^n + l_{10} \delta_{m+2}^n \tag{15}$$

$$k_1 = 1 - \frac{20\mu}{h^2}, k_2 = 26 - \frac{40\mu}{h^2}, k_3 = 66 + \frac{120\mu}{h^2}, k_4 = 26 - \frac{40\mu}{h^2},$$

$$k_5 = 1 - \frac{20\mu}{h^2}, k_6 = 1 - \frac{20\mu}{h^2}, k_7 = 26 - \frac{40\mu}{h^2}, k_8 = 66 + \frac{120\mu}{h^2},$$

$$k_9 = 26 - \frac{40\mu}{h^2}, k_{10} = 1 - \frac{20\mu}{h^2}$$

$$l_1 = 1 - \frac{20\mu}{h^2} - \frac{5z_j \Delta t}{2h}, l_2 = 26 - \frac{40\mu}{h^2} - \frac{25z_j \Delta t}{h}, l_3 = 66 + \frac{120\mu}{h^2}, l_4 = 26 - \frac{40\mu}{h^2} + \frac{25z_j \Delta t}{h},$$

$$l_5 = 1 - \frac{20\mu}{h^2} + \frac{5z_j \Delta t}{2h}, l_6 = 1 - \frac{20\mu}{h^2} + \frac{5z_j \Delta t}{2h}, l_7 = 26 - \frac{40\mu}{h^2} + \frac{25z_j \Delta t}{h}, l_8 = 66 + \frac{120\mu}{h^2},$$

$$l_9 = 26 - \frac{40\mu}{h^2} - \frac{25z_j \Delta t}{h}, l_{10} = 1 - \frac{20\mu}{h^2} - \frac{5z_j \Delta t}{2h}.$$

The systems submitted in (14) and (15) contain $(N+5)$ unknown δ_j time-parameters and $(N+1)$ equations. In each system, the imaginary parameters $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ are not within the solution region. These ones are eliminated first, due to the fact that only one solution of each system is obtained. For this reason, U and $U^{'}$ in Eq.(8) and the boundary conditions $U(x_L, t) = U(x_R, t) = 0$ and $U_x(x_L, t) = U_x(x_R, t) = 0$ are utilized. Thus, for systems (14) and (15) given above, $(N+1)$ x $(N+1)$ matrix system are obtained. The systems (14) and (15) are solved by Lie Trotter splitting algorithm and the nonlinear term z_j in Eq.(15) are applied 3 or 5 times inner iteration presented as $(\delta^*)^n = \delta^n + \frac{1}{2}(\delta^n - \delta^{n-1})$ throughout the computer run to ensure that numerical solutions are close to exact ones.

4   The initial vector δ_j^0

Let us now show how to find the initial vector δ_j^0 needed in the solution of systems (14) and (15). This initial vector δ_j^0 will be derived from the initial condition given as $U(x_j, 0) = U_N(x_j, 0) = g_0(x_j), j = 0(1)N$ and the boundary conditions. When this vector is used in numerical approximation (6) at time $t = t_0 = 0$, the following system of algebraic equations is obtained

$$
\begin{aligned}
U_m &= \delta_{j-2}^0 + 26\delta_{j-1}^0 + 66\delta_j^0 + 26\delta_{j+1}^0 + \delta_{j+2}^0, j = 0(1)N, \\
U_0 &= \delta_{-2}^0 + 26\delta_{-1}^0 + 66\delta_0^0 + 26\delta_1^0 + \delta_2^0, \\
U_1 &= \delta_{-1}^0 + 26\delta_0^0 + 66\delta_1^0 + 26\delta_2^0 + \delta_3^0, \\
&\quad . \\
&\quad . \\
&\quad . \\
U_{N-1} &= \delta_{N-3}^0 + 26\delta_{N-2}^0 + 66\delta_{N-1}^0 + 26\delta_N^0 + \delta_{N+1}^0, \\
U_N &= \delta_{N-2}^0 + 26\delta_{N-1}^0 + 66\delta_N^0 + 26\delta_{N+1}^0 + \delta_{N+2}^0.
\end{aligned}
\tag{16}
$$

As can be seen from here, the matrix system $(N+1)$ x $(N+5)$ is obtained. In order to obtain a single solution of this system, the number of unknowns and the number of equations must be equalized. For this, the parameters $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ are eliminated. To do this, the boundary conditions $U_x(x_L, t) = U_x(x_R, t) = 0$ and $U_{xx}(x_L, t) = U_{xx}(x_R, t) = 0$ are used as follows

$$
\begin{aligned}
\delta_{m-2}^0 + 26\delta_{m-1}^0 + 66\delta_m^0 + 26\delta_{m+1}^0 + \delta_{m+2}^0 &= g_0(x_m), m = 0(1)N \\
-\delta_{-2}^0 - 10\delta_{-1}^0 + 10\delta_1^0 + \delta_2^0 &= g_0^{'}(x_L), \\
\delta_{-2}^0 + 2\delta_{-1}^0 - 6\delta_0^0 + 2\delta_1^0 + \delta_2^0 &= g_0^{''}(x_L), \\
\delta_{N-2}^0 + 2\delta_{N-1}^0 - 6\delta_N^0 + 2\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0^{''}(x_R), \\
-\delta_{N-2}^0 - 10\delta_{N-1}^0 + 10\delta_{N+1}^0 + \delta_{N+2}^0 &= g_0^{'}(x_R).
\end{aligned}
\tag{17}
$$

Hence, using the equations in system (16), $(N+1)$ x $(N+1)$ dimensional matrix equation that can be solved with variant

of Thomas algorithm for the initial vector δ^0 is obtained as

$$
\begin{bmatrix}
54 & 60 & 6 & & & & & \\
25.25 & 67.5 & 26.25 & 1 & & & & \\
1 & 26 & 66 & 26 & 1 & & & \\
& & & \ddots & & & & \\
& & 1 & 26 & 66 & 26 & 1 & \\
& & & 1 & 26.25 & 67.5 & 25.25 & \\
& & & & 6 & 60 & 54 &
\end{bmatrix}
\begin{bmatrix}
\delta^0{}_0 \\
\delta^0{}_1 \\
\delta^0{}_2 \\
. \\
. \\
. \\
\delta^0{}_{N-2} \\
\delta^0{}_{N-1} \\
\delta^0{}_N
\end{bmatrix}
=
\begin{bmatrix}
U_0 \\
U_1 \\
U_2 \\
. \\
. \\
. \\
U_{N-2} \\
U_{N-1} \\
U_N
\end{bmatrix}.
$$

5 Stability Analysis

The stability analysis of the systems (14) and (15) is carried out by the Von Neumann method. Firstly, the Fourier modes, $\delta_j^n = \varrho_1^n e^{ij\Phi}$ in approximation (14) and $\Psi_j^n = \varrho_2^n e^{ij\Phi}$ in approximation (15), are substituted. in order to apply the Von Neumann method to Equation (15), it should be stated here that the moment the $3U^2$ in the term $3U^2 U_x$ is linearized, z_j will immediately behave as a local constant and hence the von Neumann method becomes applicable for the stability of the system (15). When the Euler formula $e^{i\Phi} = cos\Phi + isin\Phi$ is used, growth factors given in the following form are obtained

$$\varrho_1 = \frac{A_1 - iB_1}{A_1 + iB_1}, \quad \varrho_2 = \frac{A_1 - iC_1}{A_1 + iC_1}, \tag{18}$$

and for the expressions $k_1, k_2, ..., k_9, k_{10}$ and $l_1, l_2, ..., l_9, l_{10}$ in section 3

$$A_1 = 2a_1 - \frac{40\mu}{h^2}a_2, B_1 = 0, C_1 = \frac{5z_j \Delta t}{h}a_3$$

$$a_1 = cos2\Phi + 26cos\Phi + 33, a_2 = cos2\Phi + 2cos\Phi - 3,$$

$$a_3 = sin2\Phi + 10sin\Phi$$

$|\varrho_1| = |\varrho_2| = 1$ from Equation (18) and therefore $|\varrho_1|.|\varrho_2| = 1$. It can be clearly stated that the systems (14) and (15) are unconditionally stable. Because the conditions $|\varrho_1| \leq 1$, and $|\varrho_2| \leq 1$ are satisfied.

6 Numerical experiments and discussion

In order to demonstrate the satisfactory accuracy and performance of the algorithm applied in the present study, the error norms L_2 and L_∞, the three invariant formulas I_1, I_2, I_3 as mass, momentum and energy presented by Olver [31] and rate of convergence in both time and space pointwise in Exp.1 will be calculated by using the formulas presented in the form below for the proposed numerical scheme.

$$L_2 = ||U - U_N||_2 = \sqrt{h\sum_{j=0}^{N}(U - U_N)^2},$$

$$L_\infty = ||U - U_N||_\infty = \max_j |U - U_N|,$$

$$I_1 = \int_{x_L}^{x_R} U(x,t)dx,$$

$$I_2 = \int_{x_L}^{x_R} [U^2(x,t) + \mu U_x^2(x,t)]dx,$$

$$I_3 = \int_{x_L}^{x_R} [U^4(x,t)]dx,$$

Table 1: Analytical values of the invariants I_1, I_2, I_3 for different values of c of Example 1.

method	c	I_1	I_2	I_3
analytic	0.25	0.7853981633974	0.1666666666667	0.00520833333333
	0.50	1.5707963267949	0.6666666666667	0.08333333333333
	0.75	2.3561944901923	1.5	0.421875
	1	3.1415926535898	2.6666666666667	1.33333333333333

and The pointwise rates of convergence in space sizes Δx_m and time sizes Δt_m are computed with the following formulas respectively [18];

$$Order = \frac{log_{10}(|U^{exact} - U^{num}_{\Delta x_m}|/|U^{exact} - U^{num}_{\Delta x_{m+1}}|)}{log_{10}(\Delta x_m/\Delta x_{m+1})},$$

$$Order = \frac{log_{10}(|U^{exact} - U^{num}_{\Delta t_m}|/|U^{exact} - U^{num}_{\Delta t_{m+1}}|)}{log_{10}(\Delta t_m/\Delta t_{m+1})}.$$

For this goal, three examples will be considered.

6.1 Example I: The movement of a single solitary wave

Exact solution of the MEW equation with physical boundary conditions $U(\pm\infty) \to 0$ for a single solitary wave is presented as

$$U(x,t) = csech[k(x - x_0 - vt)]$$

and the initial condition is obtained from the exact solution for $t = 0$ as

$$U = (x,0) = csech[k(x - x_0)]$$

in which velocity of the wave is $v = \dfrac{c^2}{2}$, c is the amplitude of the wave and $k = \sqrt{\dfrac{1}{\mu}}$. Invariants I_1, I_2 and I_3 are computed as numerically and analytically respectively as follows

$$I_1 = h\sum_{j=0}^{N} U_j, I_2 = h\sum_{j=0}^{N}[U_j^2 + \mu(U_j')^2], I_3 = h\sum_{j=0}^{N} U_j^4$$

and

$$I_1 = \frac{c\pi}{k}, I_2 = \frac{2c^2}{k} + \frac{2\mu kc^2}{3}, I_3 = \frac{4c^4}{3k}.$$

In this example, in order to be able to make a comparison with some of the previous studies, the motion of a single solitary wave is presented in 12 different implementations. In these applications, the amplitude values, time increments and solution regions are different from each other. However, in all applications, the solution region $[0, 80]$ and $[0, 70]$ and the parameters $\mu = 1, x_0 = 30$ are generally chosen. The same parameter values used in previous studies given in the tables are used throughout all applications. For the amplitude value $c = 1$, the analytical solutions of these invariants are given as $I_1 = 3.1415926535898, I_2 = 2.6666666666667, I_3 = 1.3333333333333$ respectively. The computed analytical values of the invariants for different amplitudes are presented in Table 1.

Implementation 1.1 In the first application, the comparison of approximate results with those in previous studies [25]and [28] is presented in Table 2, with time increment $\Delta t = 0.05$ for the largest amplitude value $c = 1.0$. Fig.1 shows the movement of the single solitary wave and the absolute error value. At time t=20, the current error norms are 1.05×10^{-3} and 6.53×10^{-4}, lower than those in the table. The three invariant values remain almost the same.

Implementation 1.2 In the second implementation, time increment $\Delta t = 0.01$ is selected for $c = 1.0$, and the comparison of the approximate solutions with those in previous studies is shown in Table 3. At time $t = 20$, The current error norms are $4.40 \text{x} 10^{-5}$ and $2.81 \text{x} 10^{-5}$. It is clear from the table that the newly generated solutions are considerably smaller than those in [14],[17],[18] and [24] and very close to the work [28], and it is also clear that the invariants are very well preserved compared to those given in the table.

Implementation 1.3 In the third implementation, the amplitude value is taken as $c = 0.75$. As the amplitude decreases, the speed also decreases. In this case, the displacement of the wave becomes will be shorter. Tablo 4 exhibits the comparison of approximate results with those in previous studies [25] and [28] with time increment $\Delta t = 0.05$. Table 4 shows that the achieved results are quite small. The current error norms are $1.43 \text{x} 10^{-4}$ and $9.08 \text{x} 10^{-5}$ at $t = 20$. Fig.1 shows movement of the single solitary wave and the absolute error value. Three invariants are very well preserved.

Implementation 1.4 In the fourth implementation, Tablo 5 submits comparison of the approximate results with works in Refs.[14],[17],[18],[24] and [28] for amplitude value $c = 0.75$ and time increment $\Delta t = 0.01$. The current error norms are $7.01 \text{ x } 10^{-6}$ and $4.85 \text{ x } 10^{-6}$ at time $t = 20$. One can clearly see from this tablo that more excellent results than other results by applying Lie-Trotter splitting technique have been achieved. Three invariants are very well preserved.

Implementation 1.5 In the fifth implementation, the movement of the single solitary wave with the amplitude value $c = 0.5$ and time increment $\Delta t = 0.05$ is researched. As the amplitude decreases, the speed also decreases. In this situation, the displacement of the wave becomes will be shorter. The movement of the single solitary wave and the absolute error value are shown and plotted at Figure 1. The comparison of approximate results with those in Strang splitting algorithm [25] is given in Tablo 6. The current error norms are $9.20 \text{x} 10^{-6}$ and $6.40 \text{x} 10^{-6}$ at time $t = 20$. The current error norms are considerably smaller than [25] and also three invariants are very well preserved.

Implementation 1.6 In the sixth implementation, for comparison with previous studies [14],[17],[18], [24] and [28], the amplitude value $c = 0.5$ and time increment $\Delta t = 0.01$ are utilized. Comparison of newly found results with those of other methods is presented in Table 7. The current error norms are $1.18 \text{x} 10^{-6}$ and $9.5 \text{x} 10^{-7}$ at time $t = 20$. It can be clearly stated that the present results are the most satisfactory of those given in the table. Three invariants are very well preserved.

Implementation 1.7 In the seventh implementation, for comparison with studies [22],[26],[27], the solution region is taken as $[0,70]$. Comparison of the results of the present study with those of the others are given in Table 8. At time $t = 20$, the current error norms are $8.94 \text{x} 10^{-6}$ and $6.10 \text{x} 10^{-6}$. It is clear that the results of the suggested algorithm are better than the others. Moreover, all invariants are constant.

Implementation 1.8 In the eighth implementation, the smallest amplitude, that is, for the smallest speed, is chosen as $c = 0.25$. The numerical calculations of the error norms and invariants for the largest time increment $\Delta t = 0.2$ are made and the comparison of the solutions of these computations with the ones of the previous methods are given in Table 9. The current error norms are $1.20 \text{x} 10^{-6}$ and $1.01 \text{x} 10^{-6}$ at $t = 20$. It is clearly seen that the error norms produced are lower than those of the collocation [14] and the finite difference [21] and T-collocation [29] methods, so it can easily be said that there is an improvement thanks to the proposed scheme. All invariants are constant. Figure 1 displays numerical outlook of the single solitary wave and the absolute error value for $c = 0.25$.

Implementation 1.9 In the ninth implementation, to do the comparison with other studies [15], time increment $\Delta t = 0.1$ is utilized. Comparison of the gained results are done with three different variants of the finite difference method and are submitted at Table 10. The current error norms are $2.18 \text{x} 10^{-8}$ and $2.11 \text{x} 10^{-8}$ at $t = 1.0$. They are considerably smaller than previous implementations. Present invariants remain unchanged throughout the simulation.

Implementation 1.10 In the tenth implementation, Table 11 reports the comparison of existing implementations [11, 12],[14],[16–19], [21],[22–25],[28] and [29] in the literature with the new application for time increment $\Delta t = 0.05$ and amplitude $c = 0.25$. The current error norms are $2.66 \text{x} 10^{-7}$ and $3.19 \text{x} 10^{-7}$ at $t = 20$. It is clear that the results of the current algorithm are far superior to all the previous ones presented. All invariants are very well preserved. For $c = 0.25$ on $[0.80]$, the convergence orders of the present approach by calculating the error norms L_2 and L_∞ for different space step values and fixed time step value $\Delta t = 0.05$ are given at the Table 12. Also, the convergence orders of the submitted approach by calculating L_2 and L_∞ for fixed space step value $h = 0.1$ and different time step values are given at Table 13. It can be seen from the tables that the convergence orders obtained by the Lie–Trotter splitting algorithm using quintic B-spline collocation method are around 1 and 4 and so confirm the theoretical value.

Implementation 1.11 In the eleventh implementation, for comparison with existing implementations, time increment $\Delta t = 0.01$ and amplitude $c = 0.25$. are selected and comparison of the recommended one with other schemes [17],[18],[24] and [28] are reported at Table 14. The current error norms are $2.32 \text{x} 10^{-7}$ and $2.74 \text{x} 10^{-7}$ at $t = 20$. It can be said that the results found are the best of those in the table. Present invariants remain unchanged throughout the simulation.

Implementation 1.12 In the twelfth implementation, to do the comparison with other studies, time increment $\Delta t = 0.05$

Table 2: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 1.0$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	3.1415927	2.6666667	1.3333333	0.00000	0.00000
	5	3.1415926	2.6666666	1.3333333	0.27608	0.18314
	10	3.1415926	2.6666666	1.3333333	0.53590	0.33975
	15	3.1415926	2.6666665	1.3333332	0.79149	0.49613
	20	3.1415925	2.6666665	1.3333332	1.04683	0.65254
SS.Coll.[25]	20	3.1415922	2.6666685	1.3333329	2.49435	1.78185
HM(Pres.)[28]	20	3.1416760	2.6667600	1.3334280	1.50692	0.92893

Table 3: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.01$ for $c = 1.0$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	3.1415927	2.6666667	1.3333333	0.00000	0.00000
	5	3.1415927	2.6666667	1.3333333	0.01225	0.00871
	10	3.1415927	2.6666667	1.3333333	0.02299	0.01517
	15	3.1415927	2.6666667	1.3333333	0.03350	0.02162
	20	3.1415927	2.6666667	1.3333333	0.04402	0.02808
FD.[14]	20	3.1415790	2.6666350	1.3333310	0.14945	0.09870
SD.[17]	20	3.1415838	2.6666592	1.3333251	3.61948	2.25768
SD.[18]	20	3.1415860	2.6666612	1.3333279	3.61297	2.25401
DL.Coll. [24]	20	3.1415842	2.6666609	1.3333253	3.98833	2.84859
HM(Pres.)[28]	20	3.1416120	2.6667230	1.3333890	0.02601	0.01908

and amplitude $c = 0.25$. are utilized and $[0, 70]$ as solution region is taken. Comparison of the obtained results with other ones [17],[18],[24] and [28] are presented at Table 15. At time $t = 20$, the current error norms are 1.7 x 10^{-7} and 2.0x10^{-7}. It is clear that the results of the present algorithm are much better than all of the presented previous ones. All invariants are constant throughout the simulation.

6.2   Example II: Interaction of two solitary waves

In this example, Equation.1 is considered with the initial condition presented below for the interaction of two solitary waves

$$U(x, 0) = \sum_{j=1}^{2} c_j sech[k(x - x_j)]$$

in which $k = \sqrt{\dfrac{1}{\mu}}$ for $\mu = 1$. For values of the parameter $\Delta t = 0.2, h = 0.1$, the parameters $x_1 = 15, x_2 = 30$ and $c_1 = 1, c_2 = 0.5$ on $[0,80]$ are selected as used in the recent study [29]. Comparison of the invariants I_1, I_2, I_3 of example 6.2 with those in [29] at different times with the mentioned parameter values is shown in Table 16. It can be clearly observed that all the invariants from the initial to the last time throughout the run time are very well preserved according to [29]. It can be seen from Fig.2 that the smaller wave lags behind the larger one. Moreover, there was no flexible collision as the waves left small tail waves behind them after the collision. From here the truth is that these two solitary waves cannot be considered as solitons [26].

Table 4: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 0.75$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	2.3561945	1.5000000	0.4218750	0.00000	0.00000
	5	2.3561945	1.5000000	0.4218750	0.03775	0.02729
	10	2.3561945	1.5000000	0.4218750	0.07404	0.0489
	15	2.3561945	1.5000000	0.4218750	0.10889	0.0699
	20	2.3561945	1.5000000	0.4218750	0.14323	0.09075
SS.Coll.[25]	20	2.3561944	1.5000012	0.4218750	2.03819	1.46910
HM(Pres.)[28]	20	2.3562210	1.5000350	0.4218947	0.14153	0.08585

Table 5: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.01$ for $c = 0.75$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	2.3561945	1.5000000	0.4218750	0.00000	0.00000
	5	2.3561945	1.5000000	0.4218750	0.00231	0.00223
	10	2.3561945	1.5000000	0.4218750	0.00400	0.00301
	15	2.3561945	1.5000000	0.4218750	0.00552	0.00393
	20	2.3561945	1.5000000	0.4218750	0.00701	0.00485
FD.[14]	20	2.3561860	1.4999790	0.4218745	0.05193	0.03667
SD.[17]	20	2.3561893	1.4999973	0.4218732	1.48146	0.93623
SD.[18]	20	2.3561895	1.4999969	0.4218733	1.48020	0.93594
DL.Coll. [24]	20	2.3561896	1.4999982	0.4218733	2.24293	1.62010
HM(Pres.)[28]	20	2.3561560	1.4999630	0.4218543	0.01595	0.01218

Table 6: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 0.5$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	1.5707963	0.6666667	0.0833333	0.00000	0.00000
	5	1.5707963	0.6666667	0.0833333	0.00241	0.00203
	10	1.5707963	0.6666667	0.0833333	0.00475	0.00387
	15	1.5707963	0.6666667	0.0833333	0.00702	0.00516
	20	1.5707963	0.6666667	0.0833333	0.00920	0.00640
SS.Coll.[25]	20	1.5707963	0.6666673	0.0833333	1.05855	0.86024

Table 7: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.01$ for $c = 0.5$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	1.5707963	0.6666667	0.0833333	0.00000	0.00000
	5	1.5707963	0.6666667	0.0833333	0.00050	0.00060
	10	1.5707963	0.6666667	0.0833333	0.00083	0.00091
	15	1.5707963	0.6666667	0.0833333	0.00103	0.00092
	20	1.5707963	0.6666667	0.0833333	0.00118	0.00095
FD.[14]	20	1.5707920	0.6666588	0.0833333	0.01864	0.01509
SD.[17]	20	1.5707930	0.6666655	0.0833330	0.42152	0.27116
SD.[18]	20	1.5707930	0.6666653	0.0833330	0.42111	0.27119
DL.Coll. [24]	20	1.5707931	0.6666660	0.0833330	1.06979	0.86864
HM(Pres.)[28]	20	1.5708080	0.6666710	0.0833345	0.00662	0.00450

Table 8: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 0.5$ on $[0, 70]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	1.5707963	0.6666667	0.0833333	0.00000	0.00000
	5	1.5707963	0.6666667	0.0833333	0.00230	0.00182
	10	1.5707963	0.6666667	0.0833333	0.00458	0.00354
	15	1.5707963	0.6666667	0.0833333	0.00679	0.00486
	20	1.5707963	0.6666667	0.0833333	0.00894	0.00610
MQ.[22]	20	1.5707355	0.6666084	0.0833187	0.09475	0.03564
G.[22]	20	1.5707661	0.6666460	0.0833281	0.03186	0.02476
IMQ.[22]	20	1.5710582	0.6669421	0.0834022	0.25540	0.19890
Q. [22]	20	1.5705602	0.6664956	0.0832905	0.17124	0.13536
PG. [26]	20	1.57078	0.66666	0.08333	0.01172	0.00852
QBCM1.[27]	20	1.5707963	0.6666667	0.0833333	0.00920	0.00640
QBCM2. [27]	20	1.5707963	0.6666667	0.0833333	0.00920	0.00640
QBCM3. [27]	20	1.5707963	0.6666667	0.0833333	0.00860	0.00568

Table 9: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.2$ for $c = 0.25$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
LTS.Coll.	0	0.7853982	0.6666667	0.0052083	0.000000	0.000000
	5	0.7853982	0.6666667	0.0052083	0.003021	0.002348
	10	0.7853982	0.6666667	0.0052083	0.006036	0.004879
	15	0.7853982	0.6666667	0.0052083	0.009040	0.007506
	20	0.7853982	0.6666667	0.0052083	0.012027	0.010137
FD.[14]	20	0.7853977	0.1664736	0.0052083	2.701647	2.576377
Coll.[21]	20	0.7852864	0.1665818	0.0052061	2.021476	1.569539
T-Coll.[29]	20	0.7850300	0.1666259	0.0052058	1.471099	0.897036

Table 10: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.1$ for $c = 0.25$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^6	L_∞ x 10^6
LTS.Coll.	0.0	0.7853982	0.1666667	0.0052083	0.00000	0.00000
	0.2	0.7853982	0.1666667	0.0052083	0.00436	0.00418
	0.4	0.7853982	0.1666667	0.0052083	0.00873	0.00838
	0.6	0.7853982	0.1666667	0.0052083	0.01309	0.01260
	0.8	0.7853982	0.1666667	0.0052083	0.01745	0.01682
	1.0	0.7853982	0.1666667	0.0052083	0.02182	0.02106
FD.[15]1.	1.0	0.785341	0.166453	0.0052071	29.6099	23.5070
FD.[15]2.	1.0	0.787173	0.167079	0.0052460	727.1920	411.7200
FD.[15]3.	1.0	0.785398	0.166473	0.0052083	17.7110	10.8337
HM(Pres.) [28]	1.0	0.7853981	0.1666666	0.0052083	0.2706	0.2164

Table 11: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 0.25$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^4	L_∞ x 10^4
LTS.Coll.	0	0.7853982	0.1666667	0.0052083	0.00000	0.00000
	5	0.7853982	0.1666667	0.0052083	0.00071	0.00080
	10	0.7853982	0.1666667	0.0052083	0.00141	0.00163
	15	0.7853982	0.1666667	0.0052083	0.00206	0.00245
	20	0.7853982	0.1666667	0.0052083	0.00266	0.00319
LGal.[11]	20	0.7853970	0.1667636	0.0052083	0.80145	0.46009
L.Gal.[12]	20	0.7853967	0.1666663	0.0052083	0.80098	0.46061
FD.[14]	20	0.7853977	0.1664735	0.0052083	2.69281	2.56997
Multgrd.[16]	20	0.7853965	0.1666638	0.0052081	0.05208	0.05456
SD.[17]	20	0.7853967	0.1666664	0.0052083	0.51873	0.32113
SD.[18]	20	0.7853967	0.1666663	0.0052083	0.51774	0.32114
PG.[19]	20	0.7853967	0.1666663	0.0052083	0.80146	0.46121
Coll.[21]	20	0.7849545	0.1664765	0.0051995	2.90516	2.49892
D.Quad.[23]	20	0.7853979	0.1666671	0.0052084	0.01653	0.01194
DL.Coll. [24]1.	20	0.7853966	0.1666662	0.0052083	1.75277	1.76465
DL.Coll.[24]2.	20	0.7853966	0.1666662	0.0052083	1.75270	1.76459
SS.Coll.[25]	20	0.7853982	0.1666666	0.0052083	1.75081	1.76288
HM(Pres.)[28]	20	0.7853989	0.1666675	0.0052084	0.01275	0.01047
T-Coll.[29]	20	0.7850300	0.1666259	0.0052058	1.46806	0.89667

Table 12: Convergence order for Example 1 when $\Delta t = 0.05$ at $t = 20$.

h_j	$L_2 x 10^3$	$L_\infty x 10^3$	Order(L_2)	Order(L_∞)
0.8	3.06141247	2.28725768	−	−
0.4	0.19632160	0.13763922	3.9632	4.0550
0.2	0.01285990	0.00899224	3.9371	3.9364
0.1	0.00057606	0.16269906	3.9716	3.9647
0.05	0.00005149	0.00003627	3.9934	3.9896

Table 13: Convergence order for Example 1 when $h = 0.1$ at $t = 20$.

Δt_j	$L_2 x 10^3$	$L_\infty x 10^3$	Order(L_2)	Order(L_∞)
0.8	0.01754897	0.01252201	–	–
0.4	0.00876262	0.00630404	1.0020	0.9902
0.2	0.00439789	0.00315468	0.9947	0.9988
0.1	0.00220678	0.00157957	0.9950	0.9980
0.05	0.00110645	0.00079042	0.9961	0.9988

Table 14: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.01$ for $c = 0.25$ on $[0, 80]$.

method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	0.7853982	0.1666667	0.0052083	0.0000000	0.0000000
	5	0.7853982	0.1666667	0.0052083	0.0000635	0.0000705
	10	0.7853982	0.1666667	0.0052083	0.0001247	0.0001430
	15	0.7853982	0.1666667	0.0052083	0.0001814	0.0002127
	20	0.7853982	0.1666667	0.0052083	0.0002322	0.0002748
SD.[17]	20	0.7853967	0.1666664	0.0052083	0.0518107	0.0320756
SD.[18]	20	0.7853967	0.1666663	0.0052083	0.0517742	0.0321145
DL.Coll. [24]	20	0.7853967	0.1666662	0.0052083	0.1752326	0.1764220
HM(Pres.)[28]	20	0.7853976	0.1666651	0.0052082	0.0036871	0.0042041

Table 15: A comparison of invariant values and the error norms of single solitary wave with parameters $\mu = 1, \Delta t = 0.05$ for $c = 0.25$ on $[0, 70]$.

| method	t	I_1	I_2	I_3	L_2 x 10^3	L_∞ x 10^3
LTS.Coll.	0	0.7853982	0.1666667	0.0052083	0.00000	0.00000
	5	0.7853982	0.1666667	0.0052083	0.00005	0.00005
	10	0.7853982	0.1666667	0.0052083	0.00009	0.00010
	15	0.7853982	0.1666667	0.0052083	0.00013	0.00016
	20	0.7853982	0.1666667	0.0052083	0.00017	0.00020
LGal.[11]	20	0.7853970	0.1667636	0.0052083	0.08014	0.04600
MQ.[22]	20	0.7854414	0.1666622	0.0052081	0.06152	0.03324
G.[22]	20	0.00550	0.00369	0.0052080	0.03186	0.02476
IMQ.[22]	20	0.7855228	0.1667260	0.0052120	0.06676	0.05134
Q. [22]	20	0.7852686	0.1666209	0.0052055	0.05347	0.03716
PG. [26]	20	0.78539	0.16667	0.00521	0.00345	0.00203
HM(Pres.) [28]	20	0.7853975	0.1666667	0.0052083	0.00133	0.00120
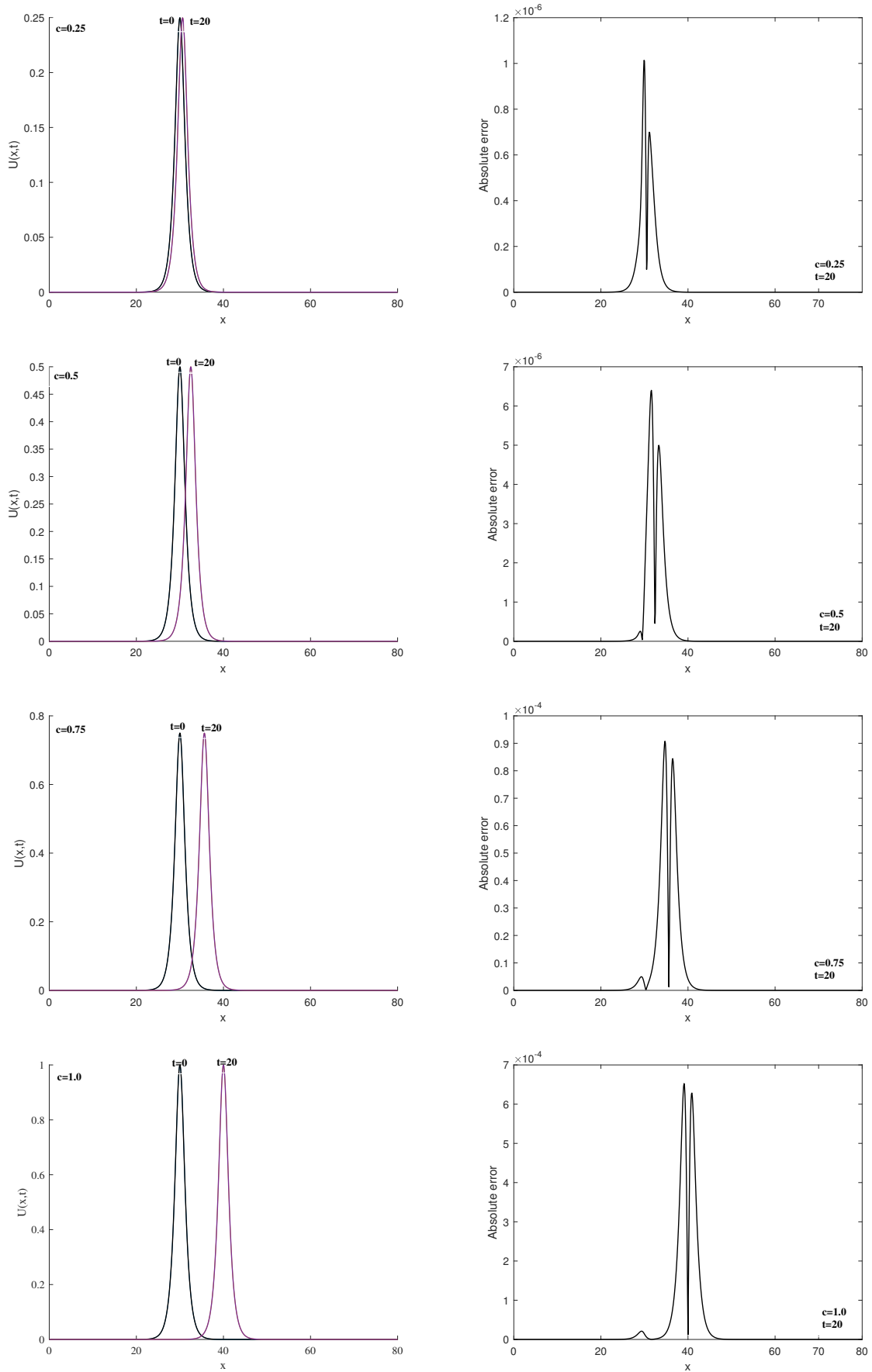
Figure 1: Movement of a single solitary wave at t=20 for different amplitude values c.
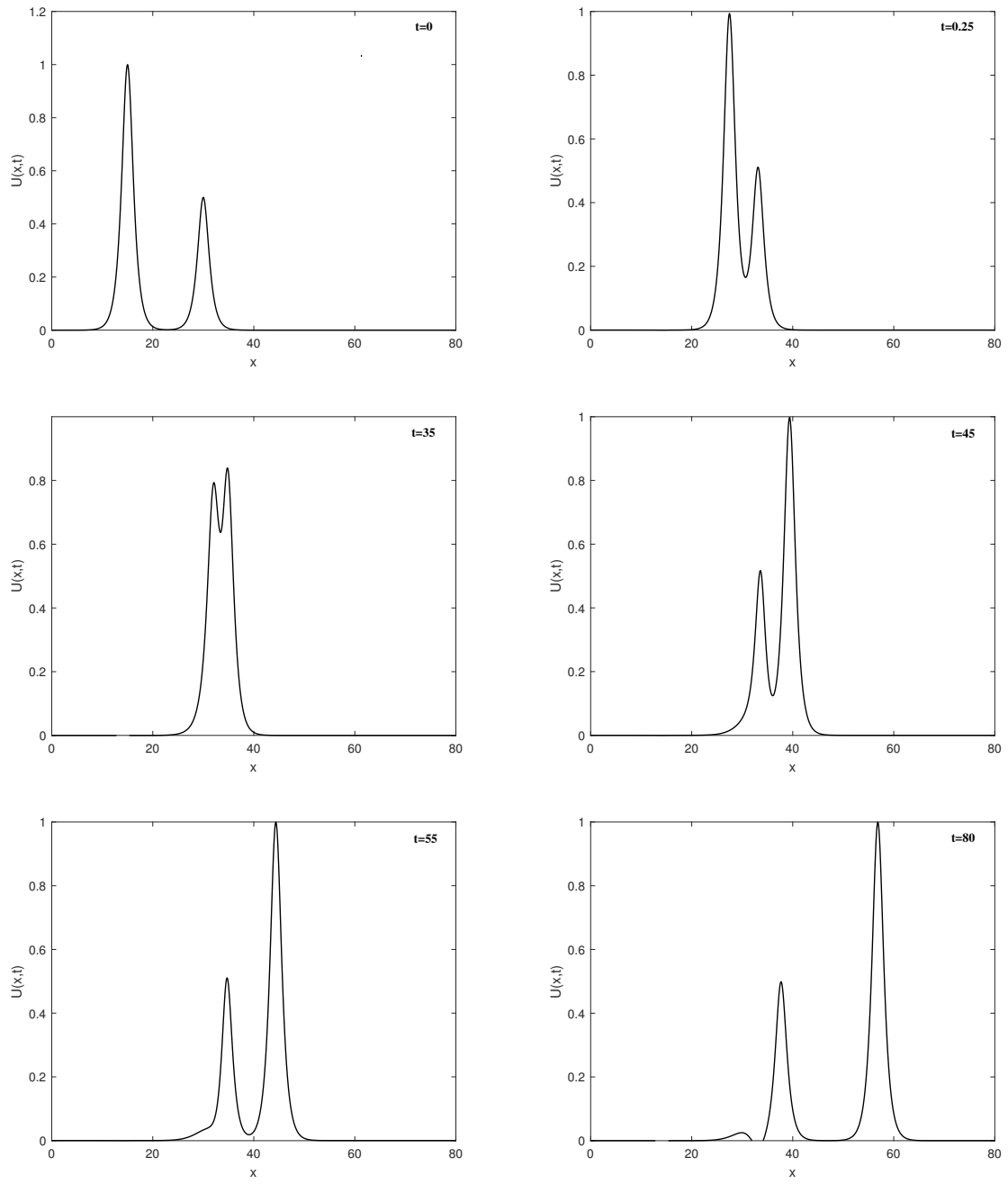
Figure 2: The interaction of two solitary waves at different times t for values of μ.

Table 16: A comparison of invariant values of interaction of two solitary waves with parameters $h = 0.1, \Delta t = 0.2$ on $[0, 80]$.

	metod			[29]		
t	I_1	I_2	I_3	I_1	I_2	I_3
0	4.712388	3.333336	1.416670	4.712388	3.333336	1.416669
10	4.712329	3.333255	1.416588	4.710180	3.331961	1.415419
20	4.712273	3.333174	1.416501	4.710180	3.331341	1.414833
30	4.712217	3.333106	1.416136	4.710181	3.329523	1.413184
40	4.712202	3.333086	1.416229	4.710181	3.329690	1.413358
50	4.712147	3.333011	1.416347	4.710180	3.330105	1.413629
55	4.712116	3.332971	1.416309	4.710180	3.329860	1.413359
60	4.712086	3.332930	1.416269	4.710180	3.329600	1.413079
70	4.712025	3.332849	1.416187	4.710180	3.329056	1.412516
80	4.711964	3.332767	1.416106	4.710180	3.328490	1.411954

6.3 Example III: Maxwellian initial condition

In this example, movement of solitary wave to check the performance of the proposed algorithm is submitted with the Maxwellian initial condition

$$U(x,0) = e^{-x^2}$$

and boundary conditions $U(x_L,t) = U(x_R,t) = 0$. For the different values $0.5, 0.1, 0.05, 0.02, 0.005, 0.0025$ of μ and the parameters $h = 0.05, \Delta t = 0.01$ on the region $[-20, 20]$ at time $T = 12.5$, Maxwell initial condition generates new solitary waves.

Implementation 2.1 In this implementation, $\mu = 1.0$ is taken for the largest μ value. The movement of the wave is illustrated in Figure 3. The values of the invariants are shown in Table 17. The invariants have retained their initial values during operation.

Implementation 2.2 In this implementation, by decreasing the value of μ, it is taken as $\mu = 0.5$. The movement of the wave is illustrated in Figure 3. The values of the invariants are exhibited in Table 17. The invariants have retained their initial values during operation.

Implementation 2.3 For this application, by reducing the value of μ to 0.1, the movement of the wave is plotted in Figure 3. The values of the invariants are shown in Table 17. The invariants have retained their initial values with little change during the run.

Implementation 2.4 For this application, the μ value decreases as $\mu = 0.05$ and the movement of the wave for this value is shown in Figure 3. The values of the invariants are listed in Table 17. It is clear that invariants have retained their initial values with little change during the run.

Implementation 2.5 In this implementation, by decreasing the value of μ, it is taken as $\mu = 0.02$. The movement of the wave is illustrated in Figure 3. The values of the invariants are shown in Table 17. The invariants have retained their initial values with little change during the run.

Implementation 2.6 For this application, By further decreasing the value of μ, it is taken as $\mu = 0.005$. The movement of the wave is displayed in Figure 3. The values of the invariants are listed in Table 17. One can clearly see that the invariants have retained their initial values with little change during the run.

7 Conclusion

In the present article, for the numerical calculations of the modified equal width wave equation, Lie-Trotter splitting algorithm combined with quintic B-spline collocation method is recommended. First of all, a wide literature review has been made about the studies done so far for the equation. Then, the Maxwellian initial condition is examined with six different applications, along with investigating the motion of a single solitary wave, which includes twelve types of

Table 17: The invariant values of Maxwellian initial condition for $h = 0.05, \Delta t = 0.01$ for different values of μ

| method	$\mu = 1.0$			$\mu = 0.5$			$\mu = 0.1$		
t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.0	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37865	0.88623
2.5	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37864	0.88625
5.0	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37864	0.88625
7.5	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37864	0.88625
10.0	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37864	0.88625
12.5	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37864	0.88625
	$\mu = 0.05$			$\mu = 0.02$			$\mu = 0.005$		
t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.0	1.77245	1.31598	0.88623	1.77245	1.27838	0.88623	1.77245	1.25958	0.88623
2.5	1.77245	1.31598	0.88631	1.77245	1.27835	0.88656	1.77211	1.25865	0.88807
5.0	1.77245	1.31597	0.88631	1.77244	1.27833	0.88654	1.77144	1.25716	0.88493
7.5	1.77245	1.31597	0.88631	1.77244	1.27832	0.88651	1.77080	1.25602	0.88255
10.0	1.77245	1.31597	0.88631	1.77243	1.27830	0.88648	1.77011	1.25437	0.87828
12.5	1.77245	1.31597	0.88631	1.77242	1.27829	0.88646	1.76948	1.25325	0.87593
Ref.[28]	$\mu = 1.0$			$\mu = 0.5$			$\mu = 0.1$		
t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.0	1.77245	2.50663	0.88623	1.77245	1.87997	0.88623	1.77245	1.37865	0.88623
2.5	1.77232	2.50645	0.88607	1.77244	1.87996	0.88622	1.77245	1.37870	0.88631
5.0	1.77242	2.50642	0.88608	1.77243	1.87995	0.88621	1.77244	1.37868	0.88629
7.5	1.77235	2.50621	0.88593	1.77248	1.88009	0.88634	1.77244	1.37869	0.88630
10.0	1.77224	2.50599	0.88576	1.77247	1.88006	0.88632	1.77244	1.37869	0.88630
12.5	1.77246	2.50635	0.88603	1.77248	1.88009	0.88635	1.77244	1.37869	0.88630
	$\mu = 0.5$			$\mu = 0.02$			$\mu = 0.005$		
t	I_1	I_2	I_3	I_1	I_2	I_3	I_1	I_2	I_3
0.0	1.77245	1.31598	0.88623	1.77245	1.27838	0.88623	1.77245	1.25958	0.88623
2.5	1.77245	1.31613	0.88654	1.77246	1.27877	0.88730	1.77246	1.25876	0.88779
5.0	1.77245	1.31610	0.88649	1.77245	1.27857	0.88691	1.77232	1.25451	0.87877
7.5	1.77245	1.31608	0.88645	1.77245	1.27838	0.88653	1.77215	1.25001	0.86825
10.0	1.77245	1.31606	0.88642	1.77245	1.27818	0.88614	1.77203	1.24635	0.86056
12.5	1.77245	1.31605	0.88640	1.77245	1.27798	0.88576	1.77186	1.24218	0.85136
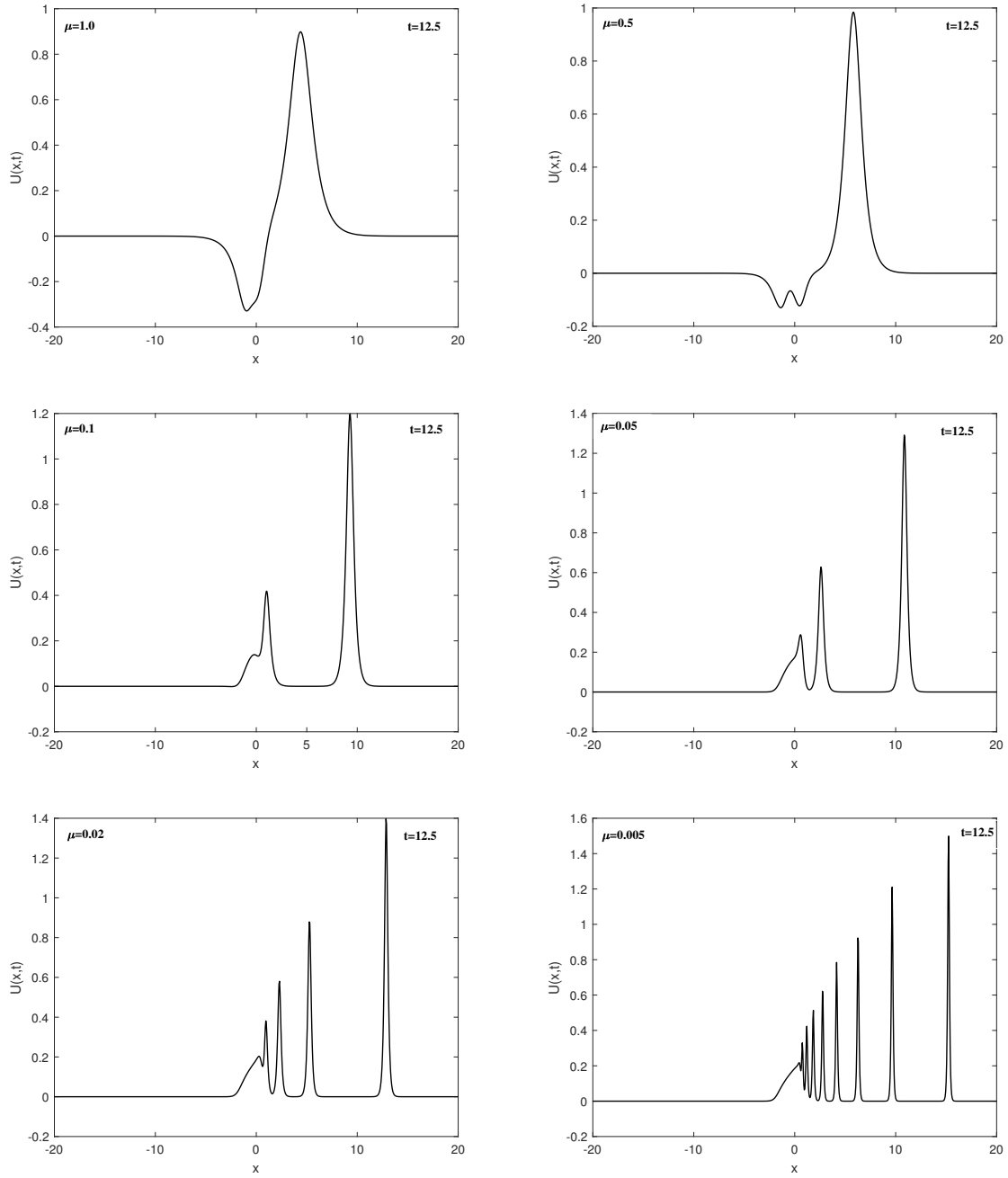
Figure 3: Maxwellian initial condition for different values of μ.

applications with different parameters. The newly produced solutions are compared with the existing applications in the literature and their behavior is depicted in graphs. As a result of those comparisons, it can be clearly seen that the proposed algorithm shows the best performance according all previous studies [11, 12], [14–19], [21–29], including those in recent years in the literature. These excellent results are new in terms of accuracy, reliability and cost compared to the results of other applications. One can easily observe this truth from the tables presented in the study. This can be meant that the algorithm presented for scientists working in this field be going to considered a useful and effective tool in obtaining approximate solutions of partial differential equations used in many fields of science.

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