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NUMERICAL TREATMENT OF NON-LOCAL BOUNDARY VALUE PROBLEM BY USING NON-STANDARD FINITE DIFFERENCE TECHNIQUE

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Abstract: This work is an attempt to solve the diffusion equation with a non-classical boundary condition by an efficient numerical technique, nonstandard finite difference (NSFD) technique. In this work a second order accurate NSFD scheme for two dimensional diffusion equation with an integral boundary condition is developed by using (1,5) explicit (forward Euler) finite difference scheme. Comparison of proposed NSFD scheme is made with the classical (1,5) finite difference (FD) scheme. Solution obtained with the proposed NSFD scheme is very close to the exact solution. Error graph verifies our claim.

Keywords: NSFD method, Diffusion equation with a non-classical boundary condition, explicit FD schemes, error analysis.

1 Introduction

Diffusion equation with initial and non-classical boundary conditions often arises in mathematical models of physics, chemistry, fluid mechanics and different branches of engineering, economics, finance [14-17] etc. Diffusion equation constrained to non-local boundary conditions occurs in many practical situations. For instance, to measure the concentration of a chemical in solution, it is feasible to utilize the absorption of light having specific frequency. Light beam is passed through a finite region of the solution having non-uniform concentration. Diffusion process helps to calculate the total mass of the chemical present in that finite region of the solution. To find the numerical solution of nonlocal diffusion equation has been emerging issue in recent years. Many researchers introduced new techniques for the solution of diffusion equation with nonlocal conditions [6-12, 22]. NSFD is a very efficient tool to solve both linear and nonlinear differential equations [1-5, 23-25]. We develop an NSFD scheme for the solution of two dimensional diffusion equation with a nonlocal boundary condition.
Consider the two dimensional diffusion equation with a nonlocal boundary condition

$$\xi_t = \xi_{xx} + \xi_{yy} \tag{1.1}$$

subject to the following initial and boundary conditions,

a) $$\xi(x,y,0) = v(x,y), \qquad 0 \le x, y \le 1, \tag{1.2}$$

b) $$\begin{cases} \xi(0,y,t) = f_0(y,t), & 0 < t \le T, \quad 0 \le y \le 1, & (1.3) \\ \xi(1,y,t) = f_1(y,t), & 0 \le t \le T, \quad 0 \le y \le 1, & (1.4) \\ \xi(x,0,t) = g_0(x)\sigma(t), & 0 \le t \le T, \quad 0 \le x \le 1, & (1.5) \\ \xi(x,1,t) = g_1(x,t), & 0 \le t \le T, \quad 0 \le x \le 1, & (1.6) \end{cases}$$

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c) $\int_0^1 \int_0^1 \xi(x,y,t)dxdy = p(t) \qquad 0 \le x,y \le 1,$ (1.7)

Here v, f_0, f_1, g_0, g_1 & p are known functions, while ξ & σ are unknown functions.

2 Numerical Technique

An NSFD scheme for ordinary and partial differential equations is developed by following rules:

i) The order of the discrete derivative and the order of continuous derivative should be equal in ordinary and partial differential equations.

ii) Step size h is replaced with the function of step size $\Phi(h) = h + O(h^2)$ provided that, $\Phi(h)$ tends to 0 as h tends to 0 and $\Psi(\tau) = \tau + O(\tau^2)$ provided that, $\Psi(\tau)$ tends to 0 as τ tends to 0.

iii) In general, nonlinear terms are transformed into nonlocal discrete representations.

iv) If a characteristic P is demonstrated by continuous model then this characteristics P preserves by numerical technique.

2.1 Nonstandard Finite Difference Scheme

We use explicit (1,5) FD scheme [20,21,26]. Initially, divide $[0,1]^2 \times [0,T]$ into $M^2 \times N$ with step sizes for space $h = \frac{1}{M}$ and time $\tau = \frac{T}{N}$.

Grid points are

$$x_l = lh, \quad l = 0,1,2,\dots,M, \tag{2.1.1}$$

$$y_m = mh, \quad l = 0,1,2,\dots,M, \tag{2.1.2}$$

$$t_n = n\tau, \quad n = 0,1,2,\dots,N, \tag{2.1.3}$$

where M is an even integer. We denote $\xi_{l,m}^n$ and σ^n as the finite difference approximations of $\xi(lh, mh, n\tau)$ and $\sigma(n\tau)$, respectively.

Here, the diffusion equation is solved by NSFD scheme and integration involved in (1.7) is solved by Simpson's numerical integration scheme [18].

The following problem

$$\xi_t = \xi_{xx} + \xi_{yy} \tag{2.1.4}$$

is solved numerically with initial values

$$\xi_{l,m}^0 = v(x_l, y_m), \qquad l,m = 0,1,2,\dots,M, \tag{2.1.5}$$

$$\xi_{0,m}^{n+1} = f_0(y_m, t_{n+1}), \qquad m = 0,1,2,\dots,M, \tag{2.1.6}$$

$$\xi_{M,m}^{n+1} = f_1(y_m, t_{n+1}), \qquad m = 0,1,2,\dots,M, \tag{2.1.7}$$

$$\xi_{l,0}^{n+1} = g_0(x_l)\sigma(t_{n+1}), \quad m = 0,1,2,\dots,M, \tag{2.1.8}$$

$$\xi_{l,M}^{n+1} = g_1(x_l, t_{n+1}), \qquad l = 0,1,2,\dots,M, \tag{2.1.9}$$

for $n = 0,1,2,\dots,N-1$, where $g_0(x)$ and $\sigma(t)$ are unknown function.

Now we develop NSFD explicit (1,5) scheme. The finite difference formulas for first order time derivative and second order space derivatives are

$$\xi_t|_{l,m}^n = \frac{\xi_{l,m}^{n+1} - \xi_{l,m}^n}{\psi(\tau)}, \tag{2.1.10}$$

$$\xi_{xx}|_{l,m}^n = \frac{\xi_{l-1,m}^n - 2\xi_{l,m}^n + \xi_{l+1,m}^n}{(\varphi(h))^2}, \tag{2.1.11}$$

$$\xi_{yy}\Big|_{l,m}^n = \frac{\xi_{l,m-1}^n - 2\xi_{l,m}^n + \xi_{l,m+1}^n}{(\varphi(h))^2}, \tag{2.1.12}$$

substituting values from (2.1.10), (2.1.11) and (2.1.12) in equation (1.1), we get

$$\frac{\xi_{l,m}^{n+1} - \xi_{l,m}^n}{\psi(\tau)} = \frac{\xi_{l-1,m}^n - 2\xi_{l,m}^n + \xi_{l+1,m}^n}{(\varphi(h))^2} + \frac{\xi_{l,m-1}^n - 2\xi_{l,m}^n + \xi_{l,m+1}^n}{(\varphi(h))^2} \tag{2.1.13}$$

$$\xi_{l,m}^{n+1} - \xi_{l,m}^n = \zeta\left(\xi_{l-1,m}^n - 2\xi_{l,m}^n + \xi_{l+1,m}^n\right) + \zeta\left(\xi_{l,m-1}^n - 2\xi_{l,m}^n + \xi_{l,m+1}^n\right) \tag{2.1.14}$$

$$\xi_{l,m}^{n+1} = \xi_{l,m}^n + \zeta\left(\xi_{l-1,m}^n - 2\xi_{l,m}^n + \xi_{l+1,m}^n\right) + \zeta\left(\xi_{l,m-1}^n - 2\xi_{l,m}^n + \xi_{l,m+1}^n\right) \tag{2.1.15}$$

$$\zeta = \frac{\psi(\tau)}{(\varphi(h))^2}$$

3 Numerical Stability of NSFD Scheme.

Stability criterion is determined for the developed NSFD scheme by applying Von-Neumann stability method, which is explained as,

$$\frac{\xi_{l,m}^{n+1}-\xi_{l,m}^{n}}{\psi(\tau)}=\frac{\xi_{l-1,m}^{n}-2\xi_{l,m}^{n}+\xi_{l+1,m}^{n}}{(\varphi(h))^2}+\frac{\xi_{l,m-1}^{n}-2\xi_{l,m}^{n}+\xi_{l,m+1}^{n}}{(\varphi(h))^2}$$

$$\xi_{l,m}^{n+1}=\xi_{l,m}^{n}+\frac{\psi(\tau)}{(\varphi(h))^2}\left(\xi_{l-1,m}^{n}-2\xi_{l,m}^{n}+\xi_{l+1,m}^{n}\right)+\frac{\psi(\tau)}{(\varphi(h))^2}\left(\xi_{l,m-1}^{n}-2\xi_{l,m}^{n}+\xi_{l,m+1}^{n}\right)$$

substituting $\xi_{l,m}^{n}$ by $\Upsilon(t)e^{i(\alpha x+\beta y)}$ in the above equation, we get,

$$\Upsilon(t+\Delta t)e^{i(\alpha x+\beta y)}=\zeta\left(\Upsilon(t)e^{i(\alpha(x-\Delta x)+\beta y)}+\Upsilon(t)e^{i(\alpha(x+\Delta x)+\beta y)}+\Upsilon(t)e^{i(\alpha x+\beta(y-\Delta y)}+\Upsilon(t)e^{i(\alpha x+\beta(y+\Delta y))}\right)+(1-4\zeta)\Upsilon(t)e^{i(\alpha x+\beta y)}$$

where,

$$\zeta=\frac{\psi(\tau)}{(\varphi(h))^2}$$

dividing both sides by $e^{i(\alpha x+\beta y)}$, we get,

$$\Upsilon(t+\Delta t)=\zeta\left(\Upsilon(t)e^{i(\alpha(-\Delta x))}+\Upsilon(t)e^{i(\alpha(\Delta x))}+\Upsilon(t)e^{i(\beta(-\Delta y)}+\Upsilon(t)e^{i(\beta(\Delta y))}\right)+(1-4\zeta)\Upsilon(t)$$

dividing both sides by $\Upsilon(t)$

$$\frac{\Upsilon(t+\Delta t)}{\Upsilon(t)}=\zeta\left(e^{i(\alpha(-\Delta x))}+e^{i(\alpha(\Delta x))}+e^{i(\beta(-\Delta y)}+e^{i(\beta(\Delta y))}\right)+(1-4\zeta)$$

$$\frac{\Upsilon(t+\Delta t)}{\Upsilon(t)}=2\zeta\left(cos(\alpha\Delta x)+cos(\beta\Delta y)\right)+(1-4\zeta)$$

after some computations, we get [19],

$$\left|\frac{\Upsilon(t+\Delta t)}{\Upsilon(t)}\right|=1-4\zeta sin^2\left(\frac{\Delta x}{2}\right)-4\zeta sin^2\left(\frac{\Delta y}{2}\right)$$

For stability, we need,

$$\left|\frac{\Upsilon(t+\Delta t)}{\Upsilon(t)}\right|\leq 1$$

In this case, stability reads when $\Delta x=\Delta y$

$$\zeta\leq\frac{1}{4},$$

where $\zeta=\frac{\psi(\tau)}{(\varphi(h))^2}$

4 Numerical Integration Procedure

In this section, we present the procedure to solve (1.7). For this, we use Simpson's rule which has fourth order truncation error. Consider

$$B(x,t)=\int_0^1\xi(x,y,t)dy \tag{4.1}$$

then

$$\int_0^1 B(x,t)dx\approx\frac{h}{3}\left(B_0+4\sum_{l=1}^{\frac{M}{2}}B_{2l-1}+2\sum_{l=1}^{\left(\frac{M}{2}\right)-1}B_{2l}+B_M\right) \tag{4.2}$$

where

$$B_l=\frac{h}{3}\left(\xi_{l,0}+4\sum_{m=1}^{\frac{M}{2}}\xi_{l,2m-1}+2\sum_{m=1}^{\left(\frac{M}{2}\right)-1}\xi_{l,2m}+\xi_{l,M}\right) \tag{4.3}$$

now let

$$C(t)=\int_0^1 B(x,t)dx \tag{4.4}$$

and we use the approximation

$$c^n=\frac{h}{3}\left(B_0^n+4\sum_{l=1}^{\frac{M}{2}}B_{2l-1}^n+2\sum_{l=1}^{\left(\frac{M}{2}\right)-1}B_{2l}^n+B_M^n\right), \tag{4.5}$$

so

$$c^n=\frac{h^2}{3}\left(\sum_{l=0}^{\left(\frac{M}{2}\right)-1}\xi_{0,0}^n+4\sum_{l=1}^{\frac{M}{2}}\xi_{2l-1,0}^n+2\sum_{l=1}^{\left(\frac{M}{2}\right)-1}\xi_{2l,0}^n+\xi_{M,0}^n\right)+S^n \tag{4.6}$$

remember that

$$c^n=\frac{h}{3}\sigma^n\int_0^1 g_0(x)dx+S^n, \tag{4.7}$$

where S^n is the summation in c^n excluding the values at the boundary $y=0$. We approximate

$$\sigma^{n+1}=\frac{p^{n+1}-S^{n+1}}{\frac{h}{3}\int_0^1 g_0(x)dx},\qquad \int_0^1 g_0(x)dx\neq 0 \tag{4.8}$$

from which the boundary values along $y=0$ may be computed at t_{n+1} using (2.1.8).

5 Application

We applied the developed NSFD scheme for the solution of two different types of problems. In first type, the value of the function $\sigma(t)$ is given in the set of boundary conditions 1.1-1.6 while in the second type of problem, the function $\sigma(t)$ is unknown. To solve this type of problem, an additional condition (1.7) is used. The procedure for the solution of both types of problems is explained in the section 2.1 and section 4.

5.1 Example

Consider the problem (1.1) – (1.7) with

$v(x, y) = \exp(x + y)$

$f_0(y, t) = \exp(y + 2t)$

$f_1(y, t) = \exp(1 + y + 2t)$

$g_1(x, t) = \exp(1 + x + 2t)$

$g_0(x) = \exp(x)$

$\sigma(t) = \exp(2t)$

$p(t) = (\exp(1) - 1)^2 \exp(2t)$

and the exact solution is

$\xi(x, y, t) = \exp(x + y + 2t)$

and the denominator functions are

$\varphi(h) = h$ and $\psi(\tau) = \exp(\tau) - 1$

Table 1.

Results for ξ with $T = 1.0, h = 0.02, \zeta = \frac{1}{6}$

			(1,5) Explicit scheme	NSFD Scheme
x	y	Exact ξ	Absolute Error	Absolute Error
0.1	0.1	9.025013	1.0566×10^{-05}	2.1091×10^{-10}
0.2	0.2	11.023176	3.1759×10^{-05}	6.3391×10^{-10}
0.3	0.3	13.463738	5.5888×10^{-05}	1.1155×10^{-09}
0.4	0.4	16.444647	7.7516×10^{-05}	1.5472×10^{-09}
0.5	0.5	20.085537	9.2114×10^{-05}	1.8386×10^{-09}
0.6	0.6	24.532530	9.6000×10^{-05}	1.9162×10^{-09}
0.7	0.7	29.964100	8.6587×10^{-05}	1.7283×10^{-09}
0.8	0.8	36.598234	6.3024×10^{-05}	1.2580×10^{-09}
0.9	0.9	44.701184	2.8318×10^{-05}	5.6522×10^{-10}

Table 1 demonstrate the absolute error between exact solution and the solution obtained by (1,5) explicit finite difference scheme and developed NSFD scheme. It reveals the efficiency of NSFD scheme.

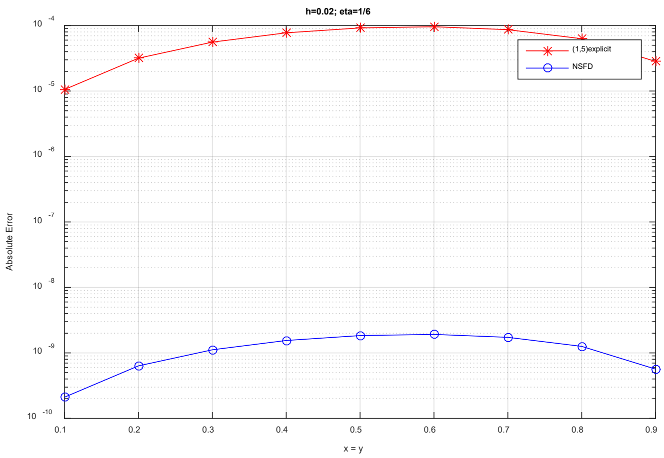

Figure 1: Error graph of (1,5) explicit and NSFD scheme

It can be noted from error graph (figure 1) that NSFD scheme is much better than (1,5) explicit scheme.

Table 2.

Results for σ with $h = 0.02, \zeta = \frac{1}{6}$

		(1,5) Explicit Scheme	NSFD Scheme
t	Exact σ	Absolute Error	Absolute Error
0.1	1.221403	1.8722×10^{-05}	1.5845×10^{-08}
0.2	1.491825	2.2969×10^{-05}	1.9222×10^{-08}
0.3	1.822119	2.8055×10^{-05}	2.3476×10^{-08}
0.4	2.225541	3.4266×10^{-05}	2.8674×10^{-08}
0.5	2.718282	4.1853×10^{-05}	3.5022×10^{-08}
0.6	3.320117	5.1119×10^{-05}	4.2776×10^{-08}
0.7	4.055200	6.2437×10^{-05}	5.2247×10^{-08}
0.8	4.953032	7.6261×10^{-05}	6.3814×10^{-08}
0.9	6.049647	9.3145×10^{-05}	7.7943×10^{-08}
1.0	7.389056	1.1377×10^{-04}	9.5201×10^{-08}

The absolute error $|\sigma_{exact} - \sigma_{approximate}|$ between developed NSFD explicit scheme and classical (1,5) FD scheme is shown in table 2

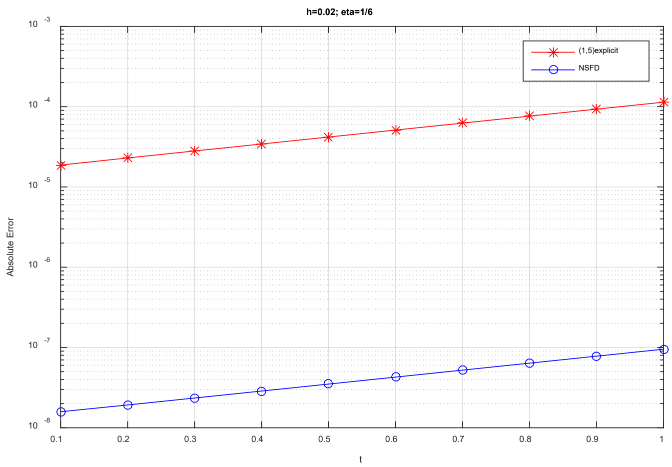

Figure 2: Error graph of (1,5) explicit and NSFD scheme for σ

Error graph of $\sigma(t)$ also represents the efficiency of NSFD scheme.

6 Conclusion

Explicit NSFD scheme is designed for the diffusion equation with one integral condition. Explicit $(1,5)$ FD schme and developed nonstandard finite difference scheme is applied to both local and nonlocal problem. Comparison between developed nonstandard finite difference scheme and $(1,5)$ explicit FD scheme show that NSFD scheme is visibly more accurate than $(1,5)$ explicit FD scheme. NSFD scheme significantly diminishes the absolute error remarkably and reduces the absolute error up to 10^5 times when local boundary conditions are used and 10^4 times when nonlocal boundary conditions are used. Stability analysis of NSFD scheme shows that it is conditionally stable.

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