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A Finite Element Method to Solve the System of Two-dimensional Burger' Equations

Selçuk Kutlay[1], Nuri Murat Yağmurlu[1] *, Yusuf Uçar[1], Orkun Taşbozan[2]
[1] Department of Mathematics, Faculty of Arts and Sciences, İnönü University, 44280 Malatya/TURKEY
[2] Department of Mathematics, Faculty of Arts and Sciences, Mustafa Kemal University, 31000, Hatay/TURKEY
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Abstract: In this paper, a Galerkin finite element method is proposed for numerically solving the system of two-dimensional Burgers' equations. The proposed method basically depends on two-dimensional Hopf-Cole transformation to convert the system of two-dimensional Burgers equations together with their initial and boundary conditions into a linear heat equation together with its corresponding initial and boundary conditions. The newly obtained linear heat equation is then solved by Galerkin finite element method using modified cubic B-spline base functions. Numerical experiments have been carried out to illustrate the applicability and efficiency of the proposed method.

Keywords: Burgers' equations; Galerkin Finite Element Method; Modified bi-cubic B-splines; Hopf-Cole transformation

1 Introduction

Burgers equation plays an important role as a model example in a wide range of physical phenomena such as fluid mechanics, gas dynamics, traffic flow, acoustics and hydrodynamics, shock waves, turbulence problems, continuous stochastic process etc. The equation is among a few widely known nonlinear partial differential equations and thus has been solved analytically for a limited set of given initial conditions. The equation was first analytically solved by Bateman [1]. He constructed the steady solution for a simple one-dimensional Burgers' equation. Then, the equation was used by J. M. Burger in [2] to model turbulence. Because of its wide spread applicability, nowadays, there exist several analytical and numerical solutions of Burgers' equation for a given set of initial and boundary conditions using various numerical schemes in the literature. But since the main focus of the present paper is to obtain the numerical solution of two-dimensional Burgers' equations, those who are interested in one-dimensional one may refer to [3]-[5].

In particular, in the present paper, we will take the following system of two-dimensional Burgers' equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad (x,y,t)\,\epsilon\, D\times(0,T] \tag{1}$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=\varepsilon\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right) \qquad (x,y,t)\,\epsilon\, D\times(0,T] \tag{2}$$

together with the initial conditions

$$u(x,y,0)=u_0(x,y), \qquad (x,y)\,\epsilon\, D \tag{3}$$
$$\upsilon(x,y,0)=\upsilon_0(x,y), \qquad (x,y)\,\epsilon\, D \tag{4}$$

boundary conditions

*Corresponding author. *murat.yagmurlu@inonu.edu.tr*

$$u(x,y,t) = f(x,y,t), \qquad (x,y,t) \,\epsilon\, \partial D \times (0,T] \tag{5}$$

$$v(x,y,t) = g(x,y,t), \qquad (x,y,t) \,\epsilon\, \partial D \times (0,T] \tag{6}$$

and the potential symmetry condition

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \tag{7}$$

into consideration on the region $D = [a,b] \times [c,d]$ with its boundary ∂D. Hopf-Cole transformation [6],[7] is a widely used analytical tool. But, it is worth to note here that while one-dimensional Hopf-Cole transformation converts one-dimensional Burgers' equation into a linear heat equation, the two-dimensional Hopf-Cole transformation can only be used when the potential symmetry condition given by Eq. (7) is satisfied by the two-dimensional Burgers' equations.

In the literature, there are several numerical methods used to solve the system of two-dimensional Burgers' equations given by 1-7. To name a few, among all others, Jain and Holla [8] have proposed two algorithms using cubic spline function technique to solve a coupled system of two-dimensional Burgers' equations. Arminjon and Beauchamp [9] have presented a finite element method on rectangular elements to obtain numerical approximations. Liao [10] has proposed a fourth-order finite difference method for solving the system of two-dimensional Burgers' equations. Zhao et al. [11] have solved the system of two-dimensional Burgers' equations by local discontinuous Galerkin finite element method. Besides these, different methods such as Chebyshev spectral collocation method [12], lattice Boltzmann method [13], fully implicit finite difference method [14], homotopy perturbation method based on the Pade approximation [15], sixth-order finite difference method [16] have also been used in the solution of two-dimensional Burgers' equations.

In this paper, first of all, we use Hopf-Cole transformation to convert the system of Burgers' equations into a linear heat equation and then use Galerkin finite element method with modified bi-cubic B-spline base functions. For this purpose, we have next modified bi-cubic b-spline functions on the boundary of a general two dimensional problem and used them to obtain numerical solutions of the heat problem, a specific form of the general two dimensional problem, by the Galerkin finite element method. In the solution process, these modified bi-cubic B-splines are used as basis functions and rectangles as element shapes.

2 Two-Dimesional Hopf-Cole Transformation

In this section, we will try to obtain the numerical solutions of the system of two-dimensional Burgers' equations given by Eqs. (1)-(7) using the Galerkin finite element method with the modified bi-cubic B-spline base functions. First of all, we introduce a new function $\phi(x,y,t)$ and use the Hopf-Cole transformations

$$u(x,y,t) = -2\varepsilon \frac{\phi_x}{\phi}, \qquad v(x,y,t) = -2\varepsilon \frac{\phi_y}{\phi}. \tag{8}$$

Using the above transformations, we obtain the following results:

$$\frac{\partial u}{\partial t} = -2\varepsilon \frac{\phi_{xt}\phi - \phi_x\phi_t}{\phi^2} \tag{9}$$

$$\frac{\partial v}{\partial t} = -2\varepsilon \frac{\phi_{yt}\phi - \phi_y\phi_t}{\phi^2} \tag{10}$$

$$\frac{\partial u}{\partial x} = -2\varepsilon \frac{\phi_{xx}\phi - \phi_x^2}{\phi^2} \tag{11}$$

$$\frac{\partial u}{\partial y} = -2\varepsilon \frac{\phi_{xy}\phi - \phi_x\phi_y}{\phi^2} \tag{12}$$

$$\frac{\partial v}{\partial x} = -2\varepsilon \frac{\phi_{xy}\phi - \phi_x\phi_y}{\phi^2} \tag{13}$$

$$\frac{\partial v}{\partial y} = -2\varepsilon \frac{\phi_{yy}\phi - \phi_y^2}{\phi^2} \tag{14}$$

$$\frac{\partial^2 u}{\partial x^2} = -2\varepsilon \left(\frac{\phi_{xxx}\phi - \phi_{xx}\phi_x}{\phi^2} - \frac{2\phi_x\phi_{xx}\phi^2 - 2\phi\phi_x^3}{\phi^4} \right) \tag{15}$$

$$\frac{\partial^2 u}{\partial y^2} = -2\varepsilon \left(\frac{\phi_{xyy}\phi - \phi_{xy}\phi_y}{\phi^2} - \frac{\phi_y\phi_{xy}\phi^2 + \phi_x\phi_{yy}\phi^2 - 2\phi\phi_x\phi_y^2}{\phi^4} \right) \tag{16}$$

$$\frac{\partial^2 v}{\partial x^2} = -2\varepsilon \left(\frac{\phi_{xxy}\phi - \phi_{xy}\phi_x}{\phi^2} - \frac{\phi_y\phi_{xx}\phi^2 + \phi_x\phi_{xy}\phi^2 - 2\phi\phi_y\phi_x^2}{\phi^4} \right) \tag{17}$$

$$\frac{\partial^2 v}{\partial y^2} = -2\varepsilon \left(\frac{\phi_{yyy}\phi - \phi_{yy}\phi_y}{\phi^2} - \frac{2\phi_y\phi_{yy}\phi^2 - 2\phi\phi_y^3}{\phi^4} \right) \tag{18}$$

If we substitute $u, v, \partial u/\partial t, \partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ into Eq. (1), we easily obtain

$$\phi_{xt}\phi - \phi_x\phi_t = \varepsilon\left(\phi\phi_{xxx} - \phi_x\phi_{xx}\right) + \varepsilon\left(\phi\phi_{xyy} - \phi_x\phi_{yy}\right). \tag{19}$$

If we divide the both sides of Eq. (19) by ϕ^2, it yields

$$\left(\frac{\phi_t}{\phi}\right)_x = \varepsilon\left[\left(\frac{\phi_{xx}}{\phi}\right)_x + \left(\frac{\phi_{yy}}{\phi}\right)_x\right] \tag{20}$$

If we integrate both sides of the newly obtained equation with respect to x, we obtain

$$\phi_t = \varepsilon\left(\phi_{xx} + \phi_{yy}\right) + \alpha_1(y,t)\phi. \tag{21}$$

Similarly, if we substitute $u, v, \partial v/\partial t, \partial v/\partial x, \partial v/\partial y, \partial^2 v/\partial x^2$ and $\partial^2 v/\partial y^2$ into Eq. (2),we easily obtain

$$\phi_{yt}\phi - \phi_y\phi_t = \varepsilon\left(\phi\phi_{xxy} - \phi_y\phi_{xx}\right) + \varepsilon\left(\phi\phi_{yyy} - \phi_y\phi_{yy}\right). \tag{22}$$

If we divide the both sides of Eq. (22) by ϕ^2, it yields

$$\left(\frac{\phi_t}{\phi}\right)_y = \varepsilon\left[\left(\frac{\phi_{xx}}{\phi}\right)_y + \left(\frac{\phi_{yy}}{\phi}\right)_y\right] \tag{23}$$

If we integrate both sides of the newly obtained equation with respect to y, we obtain

$$\phi_t = \varepsilon\left(\phi_{xx} + \phi_{yy}\right) + \alpha_2(x,t)\phi, \tag{24}$$

In Eqs. (21)-(24) $\alpha_1(y,t)$ and $\alpha_2(x,t)$ are arbitrary functions depending on x, y and t only. If we combine Eqs. (21)-(24) , we come to the conclusion that ϕ satisfies the following heat equation

$$\phi_t = \varepsilon\left(\phi_{xx} + \phi_{yy}\right) + \alpha(t)\phi, \tag{25}$$

where $\alpha(t)$ is an arbitrary function depending on only t.

Now, we can choose $\alpha(t) = 0$ in Eq. (25) to simplify the coming computations. In that case, system of two-dimensional Burgers' equations turns into the following heat equation

$$\phi_t = \varepsilon\left(\phi_{xx} + \phi_{yy}\right). \tag{26}$$

The problem of solving the system of two-dimensional Burgers' equations now has turned into the problem of solving two-dimensional heat equation. In order to be able to obtain the numerical solution for the newly obtained heat equation, we need to find out initial and boundary conditions. For this purpose we take $D = [a,b] \times [c,d]$ with $\partial D = \Gamma_1 \cup\ \Gamma_2 \cup \Gamma_3 \cup$ Γ_4 where $\Gamma_1 = \{a \le x \le b, y = c\}$, $\Gamma_2 = \{a \le x \le b, y = d\}$, $\Gamma_3 = \{x = a, c \le y \le d\}$, $\Gamma_1 = \{x = b, c \le y \le d\}$. If we use a similar way as in [10], [11], we obtain the initial and boundary conditions as follows:

$$\phi(x,y,0) = \phi(a,c,0)\exp\left(-\int_c^y \frac{v(a,s,0)}{2\varepsilon}ds - \int_a^x \frac{u(s,y,0)}{2\varepsilon}ds\right) \tag{27}$$

$$\phi(x,c,t) = \phi(a,c,t)\exp(\int_a^x -\frac{u(s,c,t)}{2\varepsilon}ds) \qquad x\,\epsilon\,[a,b] \tag{28}$$

$$\phi(x,d,t) = \phi(a,c,t)\exp(-\int_c^d \frac{v(a,s,t)}{2\varepsilon}ds - \int_a^x \frac{u(s,d,t)}{2\varepsilon}ds) \qquad x\,\epsilon\,[a,b] \tag{29}$$

$$\phi(a,y,t) = \phi(a,c,t)\exp(\int_c^y -\frac{v(a,s,t)}{2\varepsilon}ds) \qquad y\,\epsilon\,[c,d] \tag{30}$$

$$\phi(b,y,t) = \phi(a,c,t)\exp(-\int_a^b \frac{u(s,c,t)}{2\varepsilon}ds - \int_c^y \frac{v(b,s,t)}{2\varepsilon}ds) \qquad y\,\epsilon\,[c,d] \tag{31}$$

The function $\phi(a,c,t)$ is determined by following the same procedure used in [10], [11].

3 Derivation of the modified bi-cubic B-splines

3.1 Bi-cubic B-spline element

Now, we will derive modified bi-cubic B-spline base functions which are going to be used in the next section. For this purpose, the rectangular region D of the problem is subdivided into a number of uniform rectangular finite elements of sides h_x and h_y by the knots (x_i, y_j) where $0 \leq i \leq n$, $0 \leq j \leq m$. An approximation $\phi_{nm}(x,y,t)$ with cubic B-spline functions to $\phi(x,y,t)$ is taken of the form

$$\phi_{nm}(x,y,t) = \sum_{i=-1}^{n+1}\sum_{j=-1}^{m+1} \alpha_{ij}(t)B_{ij}(x,y) \tag{32}$$

where $\alpha_{ij}(t)$'s are the amplitudes of bi-cubic B-splines $B_{ij}(x,y)$ given by

$$B_{ij}(x,y) = B_i(x)B_j(y) \tag{33}$$

and $B_i(x)$ is defined as

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x-x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - 3(x-x_{i-1})^3, & x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - 3(x_{i+1}-x)^3, & x \in [x_i, x_{i+1}], \\ (x_{i+2}-x)^3, & x \in [x_{i+1}, x_{i+2}], \\ 0 & otherwise. \end{cases}$$

[18]. $B_j(y)$ can easily be found by replacing i with j and x with y. Fig.(1)depicts a region, where $h_x = h_y = 1$, so that it is divided into finite elements by the integer knots (i,j), and a single bi-cubic B-spline B_{22} which peaks on the point $(2,2)$ and also covers a total of 4x4 $= 16$ square elements. When the entire set of bi-cubic splines B_{ij}, each of which peaks on a knot (i,j), where $0 \leq i \leq 4$, $0 \leq j \leq 4$, are added to this figure, a total of 16 splines cover each finite element [19].

3.2 A modified bi-cubic B-spline element

To show how to modify bi-cubic spline functions on the boundary, we consider the two-dimensional general linear equation of the form

$$\frac{\partial\phi}{\partial t} = a(x,y,t)\frac{\partial^2\phi}{\partial x^2} + b(x,y,t)\frac{\partial^2\phi}{\partial y^2} + c(x,y,t)\frac{\partial\phi}{\partial x} + d(x,y,t)\frac{\partial\phi}{\partial y} + e(x,y,t)\phi + f(x,y,t) \tag{34}$$

subject to the initial condition

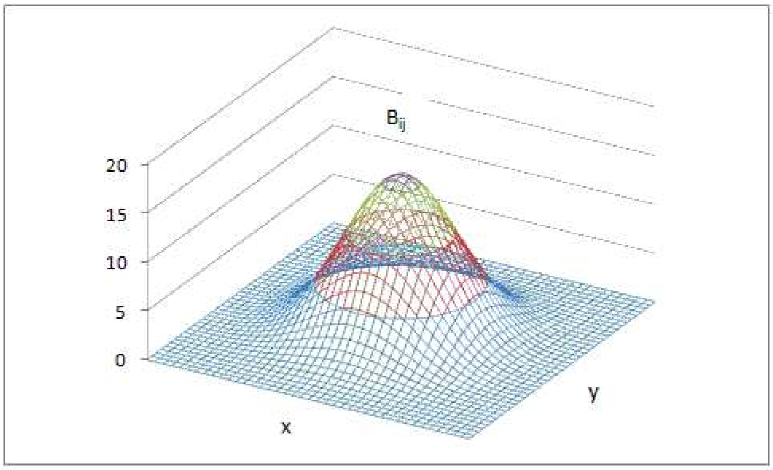

Figure 1: The bi-cubic B-spline B_{22} centered on (2,2) covering 16 finite elements of side 1.

$$\phi(x,y,0)=\phi_0(x,y) \tag{35}$$

and boundary conditions

$$\phi(x,y_0,t)=g_1(x,t), \qquad x_0\le x\le x_n \tag{36}$$

$$\phi(x,y_m,t)=g_2(x,t), \qquad x_0\le x\le x_n \tag{37}$$

$$\phi(x_0,y,t)=h_1(y,t), \qquad y_0\le y\le x_m \tag{38}$$

$$\phi(x_n,y,t)=h_2(y,t), \qquad y_0\le y\le x_m \tag{39}$$

where $x\in[x_0,x_n], y\in[y_0,y_m]$ and $g_1(x,t)$, $g_2(x,t)$, $h_1(y,t)$, $h_2(y,t)$ are given functions.

Now, it is supposed that both the x-space variable domain and y-space variable domain of the system (34)-(39) are divided into n and m sub-intervals, respectively, by the set of $n+1$ distinct grid points x_i $(i=0(1)n)$ and $m+1$ distinct grid points y_j $(j=0(1)m)$ such that

$$0=x_0<x_1<...<x_n=1 \qquad \text{and} \qquad 0=y_0<y_1<...<y_m=1.$$

Since a cubic B-spline function covers four consecutive elements, we add six additional grid points x_{-3},x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3} in the x-direction and ten additional grid points y_{-3}, y_{-2}, y_{-1}, y_{m+1}, y_{m+2}, y_{m+3} in the y-direction. To find an approximate solution in the form of Eq. (32) to the problem given by Eqs. (34)-(39) with the Galerkin method, we do need to redefine the basis functions into a new set of basis functions which all vanish on ∂D. The redefining process of the basis functions is done and the general approximation $\phi_{nm}(x,y,t)$ to $u(x,y,t)$ has been obtained in the following form

$$\phi_{nm}(x,y,t)=\frac{\varphi_1(x,y,t)+\varphi_2(x,y,t)}{2}+\sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_{ij}(t)\widetilde{B}_i(x)\widetilde{B}_j(y) \tag{40}$$

$$=\varphi(x,y,t)+\sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_{ij}(t)\widetilde{B}_i(x)\widetilde{B}_j(y)$$

where the new set of basis functions are $\widetilde{B}_i(x)\widetilde{B}_j(y)$ for $i=0(1)n$, $j=0(1)m$, which all vanish on ∂D and $\varphi(x,y,t)$ given in Eq. (40) satisfies the boundary conditions given by Eqs. (36)-(39). The profiles of the modified cubic B-splines are shown in Fig. (2) for 4 elements.

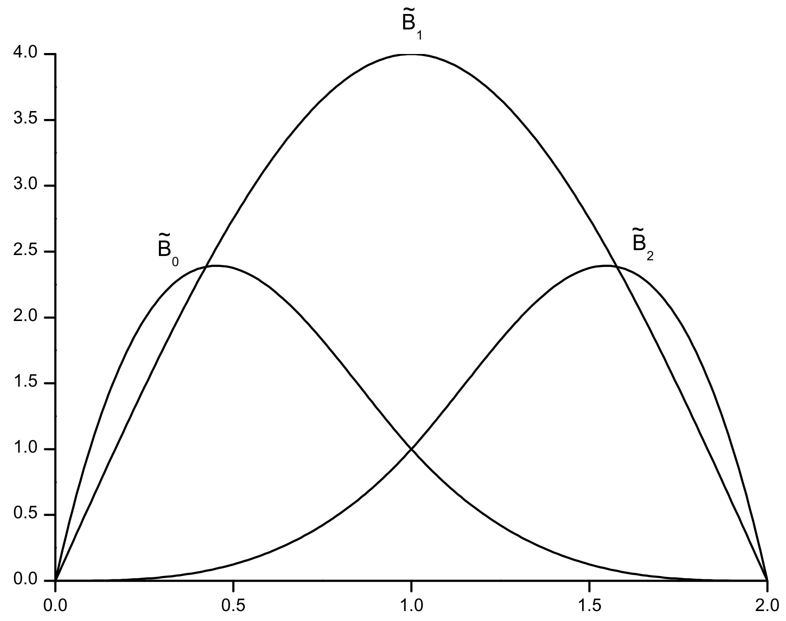

Figure 2: Modified cubic B-spline functions $\widetilde{B}_i(x)$, $i = 0(1)2$.

4 Galerkin Finite Element Solutions of the Model Problem

If we take

$$a(x, y, t) = b(x, y, t) = \varepsilon$$

and

$$c(x, y, t) = d(x, y, t) = e(x, y, t) = f(x, y, t) = 0$$

in Eq. (34), we easily obtain the heat equation. Now it is time to obtain the weak form of the equation. For this purpose, all terms in Eq. (25) are taken to the right hand side of the equation and then multiplied by the weight function $\Psi(x, y)$. Finally, by integrating the resulting equation over the region D and setting it to zero, we get

$$\iint\limits_D \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - \varepsilon \frac{\partial \phi}{\partial t} \right\} \Psi dxdy = 0 \tag{41}$$

where $\Psi = \widetilde{B}_k(x)\widetilde{B}_l(y)$ for $k = 0(1)n$ and $l = 0(1)m$. By applying the Green Theorem (see, e.g. Reddy [21]) to Eq. (41), the weak form of the model problem in the global coordinate system is obtained as follows

$$\iint\limits_D \left\{ \varepsilon \frac{\partial \phi}{\partial t} \Psi + \frac{\partial \phi}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \Psi}{\partial y} \right\} dxdy = 0. \tag{42}$$

To change from the global coordinate system into the local one, we use the transformations $h_x \xi = x - x_n$ and $h_y \eta = y - y_m$. Thus, the weak form (42) transforms to the form

$$\varepsilon h^2 \int_0^1 \int_0^1 \frac{\partial \phi}{\partial t} \Psi d\xi d\eta + \int_0^1 \int_0^1 \left(\frac{\partial \phi}{\partial \xi} \frac{\partial \Psi}{\partial \xi} + \frac{\partial \phi}{\partial \eta} \frac{\partial \Psi}{\partial \eta} \right) d\xi d\eta = 0. \tag{43}$$

4.1 Application of the method

In this subsection, the numerical solutions of the model problem are obtained by the Galerkin finite element method using the modified bi-cubic B-spline basis functions. The numerical scheme is implemented by dividing the region

$D = [0,1] \times [0,1]$ into 100 ($h_x = h_y = 1/10$) elements. In all numerical calculations, the coefficient ε in Eq. (26) is taken as 10^{-2}.

For this, the approximate solution $\phi_{nm}(\xi, \eta, t)$ for each element is written in the weak form (43) and then a coefficient matrix is obtained for each element. By combining the coefficient matrix for each element, we obtain an algebraic equation in the form [17]

$$ZA(0) = g \tag{44}$$

and an iterative equation in the form

$$G\frac{dA(t)}{dt} = PA(t) \tag{45}$$

$$\left[G - \frac{1}{2}\Delta t\ P\right] A(t_{r+1}) = \left[G + \frac{1}{2}\Delta t\ P\right] A(t_r) \tag{46}$$

$$G = [g_{i_l j_l}], \quad g_{i_k j_l} = h^2 \int_0^1\int_0^1 \widetilde{B}_i(\xi)\widetilde{B}_j(\eta)\widetilde{B}_k(\xi)\widetilde{B}_l(\eta)d\eta d\xi \tag{47}$$

$$P = [p_{i_l j_l}], \quad p_{i_l j_l} = -\int_0^1\int_0^1 \frac{\partial \widetilde{B}_i(\xi)}{\partial \xi}\widetilde{B}_j(\eta)\frac{\partial \widetilde{B}_k(\xi)}{\partial \xi}\widetilde{B}_l(\eta)d\eta d\xi - \tag{48a}$$

$$\int_0^1\int_0^1 \widetilde{B}_i(\xi)\frac{\partial \widetilde{B}_j(\eta)}{\partial \eta}\widetilde{B}_k(\xi)\frac{\partial \widetilde{B}_l(\eta)}{\partial \eta}d\eta d\xi$$

$$A_s(t) = [A_1(t)\ A_2(t)\ ...\ A_{(m+3)(n+3)}(t)]^t; \tag{49}$$

where $i, j, k, l = 0(1)n$, $i_l = j(n+1) + i$ and $j_l = l(n+1) + k$

$$Z = [z_{i_l j_l}], \qquad z_{i_l j_l} = \int_0^1\int_0^1 \widetilde{B}_i(\xi)\widetilde{B}_j(\eta)\widetilde{B}_k(\xi)\widetilde{B}_l(\eta)d\eta d\xi \tag{50}$$

where $i, j, k, l = 0(1)n$, $i_l = j(n+1) + 1$ and $j_l = l(n+1) + k$

$$g = [g_{i_l}], \qquad g_{i_l} = \int_0^1\int_0^1 u_0(\xi,\eta)\widetilde{B}_i(\xi)\widetilde{B}_j(\eta)d\eta d\xi. \tag{51}$$

where $i, j = 0(1)1$, $i_l = j(n+1) + 1$

Example 1 ([10], [11]) In this example, we consider the system of two-dimensional Burgers' equations given in Eqs. 1-2 over a square domain $D : [0,1] \times [0,1]$ with the initial conditions

$$u(x,y,0) = \frac{-4\varepsilon\pi\ \cos(2\pi x)\sin(\pi y)}{2 + \sin(2\pi x)\sin(\pi y)}, \qquad (x,y)\ \epsilon\ D,$$

$$v(x,y,0) = \frac{-2\varepsilon\pi\ \sin(2\pi x)\cos(\pi y)}{2 + \sin(2\pi x)\sin(\pi y)}, \qquad (x,y)\ \epsilon\ D,$$

and boundary conditions

$$u(0,y,t) = -2\varepsilon\pi\ e^{-5\pi^2\varepsilon t}\sin(\pi y) \quad t \geq 0,$$

$$u(1,y,t) = -2\varepsilon\pi\ e^{-5\pi^2\varepsilon t}\sin(\pi y) \quad t \geq 0,$$

$$u(x,0,t) = 0 \quad t \geq 0,$$

Table 1: Numerical results of Example 1 at $T = 1$ for $\varepsilon = 0.1$ and $h_x = h_y = 0.1$.

Δt	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{160}$
$E_u(\Delta t)$	$4.4950-04$	$1.6781-04$	$9.9787-05$	$8.2727-05$	$7.8459-05$
$r_u = \frac{E_u(\Delta t)}{E_u(\frac{\Delta t}{2})}$	$-$	2.6787	1.6817	1.2062	1.0544
$E_v(\Delta t)$	$2.5260-04$	$1.1088-04$	$7.8874-05$	$7.4169-05$	$7.4169-05$
$r_v = \frac{E_v(\Delta t)}{E_v(\frac{\Delta t}{2})}$	$-$	2.2781	1.4058	1.0634	1.000

$$u(x,1,t) = 0 \quad t \geq 0,$$

$$v(0,y,t) = 0 \quad t \geq 0,$$

$$v(1,y,t) = 0 \quad t \geq 0,$$

$$v(x,0,t) = -\varepsilon\pi\, e^{-5\pi^2\varepsilon t}\sin(2\pi x) \quad t \geq 0,$$

$$v(x,1,t) = \varepsilon\pi\, e^{-5\pi^2\varepsilon t}\sin(2\pi x) \quad t \geq 0,$$

for which the exact solutions are

$$u(x,y,t) = -2\varepsilon\frac{2\pi\, e^{-5\pi^2\varepsilon t}\cos(2\pi x)\sin(\pi y)}{2+\ e^{-5\pi^2\varepsilon t}\sin(2\pi x)\sin(\pi y)}, \qquad (x,y)\,\epsilon\, D,$$

$$v(x,y,t) = -2\varepsilon\frac{\pi\, e^{-5\pi^2\varepsilon t}\sin(2\pi x)\cos(\pi y)}{2+\ e^{-5\pi^2\varepsilon t}\sin(2\pi x)\sin(\pi y)}, \qquad (x,y)\,\epsilon\, D,$$

If the solution domain D of the problem is equally divided into $10 \times 10 = 100$ elements, 100 squares having the sides of $h_x = h_y = h = 1/10$ are obtained. If the approximate solution $\phi_{nm}(\xi,\eta,t)$ is constructed for each element, a total number of $11 \times 11 = 121$ global element parameters over the region D are obtained depending on the local element parameters $\alpha_i(t)$ and $\beta_j(t)$, $(i,j = -1(1)10)$. It is obvious that we need to find the global element parameters $A_i(t), (i = 1(1)121)$ in order to obtain the approximate solution $\phi_{nm}(\xi,\eta,t)$ for each element. By solving these algebraic equations, element parameters $A_i(t), (i = 1(1)121)$ are obtained at times $t = 0.0$ and $t = 1.0$ for different values of Δt. The obtained element parameters are put in their places in the element equations, and then the approximate solution $\phi_{nm}(\xi,\eta,t)$ for each element at times $t = 0.0$ and $t = 1.0$ is found. Finally, using Hopf-Cole transformations given in (8), we obtain the numerical approximations of $u(x,y,t)$ and $v(x,y,t)$.

In order to measure how good the numerical solutions obtained by the Galerkin finite element method with the bi-cubic B-spline basis functions, the error norm $E_u(\Delta t)$ defined as

$$E_u(\Delta t) = \|u - u_{nm}\|_\infty = \max_{0<i<n_{i_p}} |u_i - (u_{nm})_i|$$

is computed and tabulated in Table 1. As seen from the table, the approximate solutions become better as Δt decreases. Moreover the ratios of errors $r_u = E_u(\Delta t)/E_u(\Delta t/2)$ and $r_v = E_v(\Delta t)/E_v(\Delta t/2)$ have also been computed and tabulated. Physically they show how more accuarate results are obtained by using finer mesh sizes.

Example 2 ([10], [11]) As a second example, we consider the two-dimensional Burgers' equations of the following form

$$\frac{\partial u}{\partial t} + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \varepsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (x,y,t)\epsilon D\times(0,T]$$

Table 2: Numerical results of Example 2 at $T = 1$ for $\varepsilon = 0.1$ and $h_x = h_y = 0.1$.

Δt	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$	$\frac{1}{160}$
$E_u(\Delta t)$	$4.388-02$	$3.602-02$	$3.510-2$	$6.491-03$	$3.480-03$

over a square domain $D : [0,1] \times [0,1]$ with the initial condition

$$u(x,y,0) = \frac{1}{1+e^{(x+y)/2\varepsilon}}, \qquad (x,y)\,\epsilon\, D,$$

and boundary conditions

$$u(0,y,t) = \frac{1}{1+e^{(y-t)/2\varepsilon}},\;\; y \in [0,1], \quad t > 0,$$

$$u(1,y,t) = \frac{1}{1+e^{(1+y-t)/2\varepsilon}},\;\; y \in [0,1], \quad t > 0,$$

$$u(x,0,t) = \frac{1}{1+e^{(x-t)/2\varepsilon}},\;\; x \in [0,1], \quad t > 0,$$

$$u(x,1,t) = \frac{1}{1+e^{(1+x-t)/2\varepsilon}},\;\; x \in [0,1], \quad t > 0,$$

for which the exact solution is

$$u(x,y,t) = \frac{1}{1+e^{(x+y-t)/2\varepsilon}}, \qquad (x,y)\,\epsilon\, D, \quad t \geq 0.$$

But, although Eqs. 1-2 are the coupled system of Burgers' equations, in this example, we have only one equation and only one unknown function $u(x,y,t)$. Thus to be able to use the currently presented method in this paper, we definitely need an auxiliary function $v(x,y,t) = u(x,y,t)$. By using this auxiliary function $v(x,y,t)$ we obtain a system of equations similar to those given in Eqs. (1)-(2). The potential symmetry condition is automatically satisfied as we have taken $v(x,y,t) = u(x,y,t)$.

Again using the error norm L_∞ defined previously, we have compared the approximate values in Table 2. As it is seen from the table, the approximate solutions become better as Δt decreases.

5 Conclusions

In this paper, a modified bi-quintic B-spline Galerkin finite element method is proposed and successfully applied to the system of two-dimensional Burgers' equations to obtain its numerical solutions. The agreement between our numerical results and the exact solution is satisfactorily good. The obtained numerical results showed that the present method is a remarkably successful numerical technique and can also be applied to a large number of physically important two dimensional non-linear problems.

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